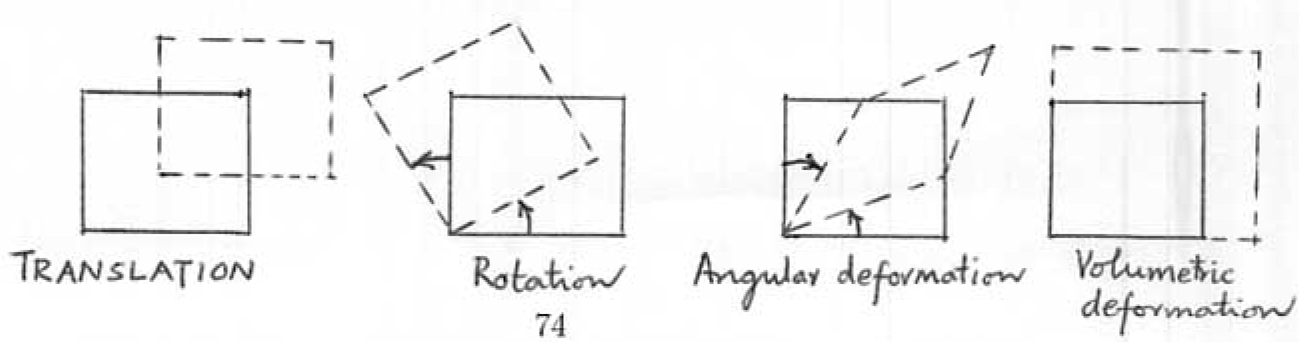
6 Chapter 6: Differential Analysis of Fluid Flow

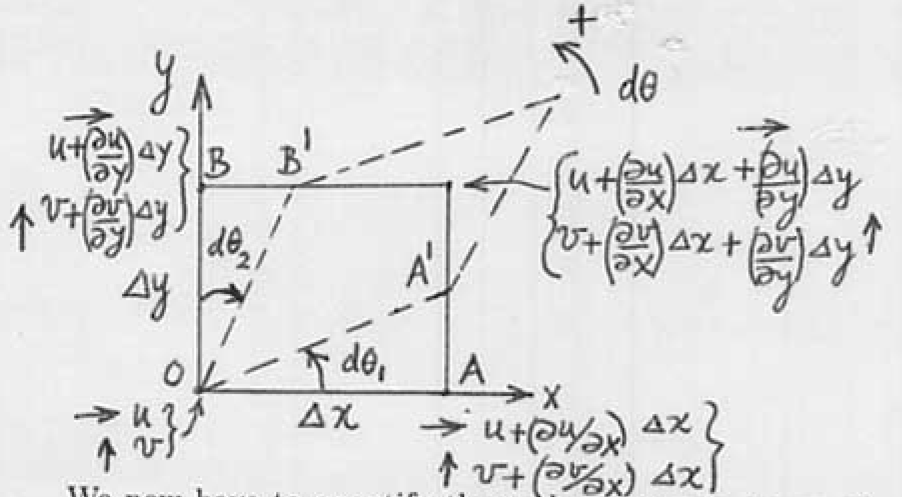
When details of fluid flow are important, we use differential analysis. We look at an infinitesimal cv as different from a finite cv that we used in our integral treatment.

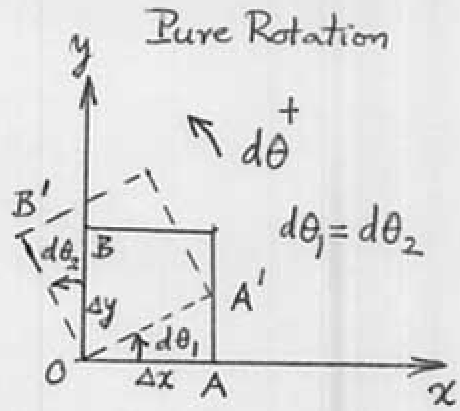
6.1 Detailed kinematics of fluid flow

Consider a very small fluid element ($\Delta x \; \Delta y \; \Delta z$) in a flow (say, a Lagrangian observation). Let the flow have some finite viscosity so that shear forces are also present in addition to pressure and body forces. As the element moves form one location to another, not only does the element occupy a new location but its velocity may be very different (the velocity components may be very different from those at the previous location). The pressure at the new location may be very different from the value at the previous location. The observed effects cannot be explained by a simple translation of the element and we have to ascribe the changes as being due to various stresses experienced by the fluid element along its path. A mere translation of a fluid element cannot result in a stress, since by a change of coordinate system the element can be brought to rest. Similarly, a uniform rotation or solid body rotation of a fluid element cannot result in a stress since, with respect to the rotating system, there is no relative motion. Thus the effects of stresses that we observe in a fluid flow must be the result of distortion or deformation of the fluid element. These distortions may be volumetric associated with linear deformations in coordinate directions as well as angular deformations associated with shape deformation in an angular sense. So to describe the motion of the fluid element, in addition to **Translation**, we have to consider **Rotation**, **Angular deformation**, and **Volumetric deformation**.

Translation is linear displacement from location, say, (x_0, y_0, z_0) to (x_1, y_1, z_1). If there are no velocity gradients at all in a flow, then a fluid element in motion will simply translate from one location to another. If there are no velocity gradients at all, then there are no accompanying stresses in the flow field. The fluid element in such a case will retain its shape. In **Rotation**, the orientation of the element as shown in the figure, where the sides of the element are parallel to the coordinate axes, may change about any one (or all three) of the coordinate axes. Again there is no distortion in shape. **Angular deformation** involves a distortion of the element in which planes of the element that were originally perpendicular are no longer perpendicular. **Volumetric deformation** involves a change in shape without a change in the orientation of the element and here, planes of the element that were originally perpendicular remain perpendicular.







We now have to quantify these phenomena in terms of measureable flow quantities such as velocity. Eventually relate the effects of these phenomena to applied forces whcih cause the motion. Otherwise we cannot analytically study the flow. Translation of a fluid element is easy to understand and note. The others are not straightforward.

6.1.1 Rotation, Angular deformation, and Volumetric deformation

A fluid element may undergo rotation due to angular momentum in the flow field. For a solid body, we measure rotation by noting the angular displacement of a line on the plane of rotation from a reference. However, a fluid element is deformable and therefore to measure rotation, we choose to consider the **the average** rotation of two lines of the element that were mutually perpendicular at the beginning of the flow. Figures (a) and (b) shows a plane fluid element of sides Δx and Δy that lies on the $x-y$ plane. Consider Figure (a). The axis of rotation is the z axis and OA and OB are the two initially perpendicular lines on the fluid element. The velocity components at O are u and v. These are increased at A and B to quantities expressible through Taylor's series and as indicated in the figure. The OA and OB shall move on to the positions OA' and OB' owing to the net velocity differences at A and B over the components at O. These constitute the angular deformation of the fluid element. Over a time dt the displacements are

$$AA' = \left(\frac{\partial v}{\partial x}\Delta x\right) dt \tag{292}$$

$$BB' = \left(\frac{\partial u}{\partial y}\Delta y\right) dt \tag{293}$$

The corresponding angles of deformation per unit time are obtained by dividing the respective arm lengths. Therefore,

$$d\dot{\theta}_1 = \lim_{dt\to 0}\frac{d\theta_1}{dt} = \lim_{dt\to 0}\frac{AA'/\Delta x}{dt} = \lim_{dt\to 0}\frac{\left(\frac{\partial v}{\partial x}\Delta x\right) dt}{\Delta x\, dt} = \frac{\partial v}{\partial x}, (\text{ anti} - \text{clockwise}), \tag{294}$$

$$\text{Similarly, } d\dot{\theta}_2 = \frac{\partial u}{\partial y}, (\text{ clockwise}). \tag{295}$$

We now adopt a convention. Anti-clockwise rotation is positive. The average angular velocity of the fluid element about the z axis in Figure (a) is

$$\omega_z = \frac{1}{2}\left(d\dot{\theta}_1 + d\dot{\theta}_2\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

For a three dimensional element, the rotations about the x and y axis are similarly obtained as, and we list the three as:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (296)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad (297)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad (298)$$

$$\text{and}, \omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \quad (299)$$

In pure rotation, the fluid element, as shown in Figure (b), will rotate about the z axis as an undeformed element such that $d\dot{\theta}_1 = d\dot{\theta}_2$. (Forced vortex). But, for pure rotation, $d\dot{\theta}_2 = -\frac{\partial u}{\partial y}$ and it is anticlockwise. Therefore, for pure rotation, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, and $\omega_z = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Next, let us look at general angular deformation. We see from the Figure (a) that, in addition to the rotation associated with derivatives $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$, these derivatives may also cause the element to undergo an angular deformation and hence a change in shape. The change in the right angle formed by the lines OA and OB due to such angular deformation, $(d\theta_1 - d\theta_2)$, is called the **shearing strain** in the $x - y$ plane, γ_{xy}. The rate of angular deformation is the rate of decrease of the angle between lines OA and OB. $\left(d\dot{\theta}_1 - d\dot{\theta}_2\right)$ is called the **rate of shearing strain** or **the rate of angular deformation**, $\dot{\gamma}_{xy}$. Thus,

$$\dot{\gamma}_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \epsilon_{xy} = \epsilon_{yx}, \quad (300)$$

where the ϵ notation has been included since many people use that instead of $\dot{\gamma}$ to denote rate of shearing strain. Similarly,

$$\dot{\gamma}_{yz} = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \epsilon_{yz} = \epsilon_{zy}, \quad (301)$$

$$\dot{\gamma}_{zx} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \epsilon_{zx} = \epsilon_{xz}. \quad (302)$$

- We see that angular deformations are associated with shear strain rates. We would expect the shear strain rates to arise as a result of shear stresses.

Clearly, the rate of shearing strain is seen to be zero for pure rotation (for example, solid body rotation).

6.1.2 Vorticity, Irrotational Flow

We note from Vector calculus that, $2\ \omega = \nabla \times \mathbf{V}$. It is customary to set $2\ \omega = \nabla \times \mathbf{V} = \zeta$, and ζ is called the **Vorticity** of the flow. When $\zeta = 0$, the flow is said to be **Irrotational**. Therefore, for an irrotational flow,

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{303}$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \tag{304}$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \tag{305}$$

Irrotationality of a two-dimensional flow in the $x - y$ plane would mean:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \tag{306}$$

This could be satisfied if (1) $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0$ or if (2) $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$. Condition (2) is satisfied when (1) is true. Condition (1) is more stringent, and is satisfied in a rectilinear motion of an inviscid fluid. When an inviscid fluid is following in a curvilinear path, condition (1) is not satisfied but condition (2) may hold (Free vortex).

- Volumetric deformation

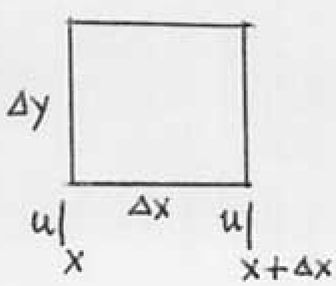
During linear deformation, the shape of the fluid element, described by the angles at its vertices, remains unchanged, since right angles continue to be right angles. The fluid element will change length in the x direction only if $\frac{\partial u}{\partial x}$ is non-zero. Similarly for changes in the y and z dimensions, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ must both be non-zeros. But, $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ are the components of longitudinal rates of strain in the x, y and z directions, respectively. This can be seen as follows: Define,

$$\text{Rate of Volumetric strain} = \frac{\text{Rate of change of Volume}}{\text{Volume}} \tag{307}$$

$$\frac{1}{\Delta v}\frac{d(\Delta v)}{dt} = \frac{1}{\Delta v}\frac{d(\Delta x\ \Delta y\ \Delta z)}{dt} \tag{308}$$

$$\frac{1}{\Delta v}\frac{d(\Delta x\ \Delta y\ \Delta z)}{dt} = \frac{1}{\Delta v}\{\Delta y \Delta z \frac{d\ \Delta x}{dt} + ...\} \tag{309}$$

But, consider for example, the x direction. From Taylor's series,



$$u|_{x+\Delta x} - u|_x = \frac{\partial u}{\partial x}\Delta x \tag{310}$$

$$\text{Also, } u|_{x+\Delta x} - u|_x = \frac{d(\Delta x)}{dt} \tag{311}$$

$$\text{Therefore, } \frac{d(\Delta x)}{dt} = \frac{\partial u}{\partial x}\Delta x \tag{312}$$

Similar terms will apply for the y and z directions. Thus, from Equations 309 and 312,

$$\text{Rate of Volumetric strain } = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V} \tag{313}$$

The quantities $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$ cause a linear deformation in a fluid element. We see that these derivatives denote rates of strain in normal directions or normal strain rates. Formally, we set

$$2\left(\frac{\partial u}{\partial x}\right) = \epsilon_{xx}, \tag{314}$$

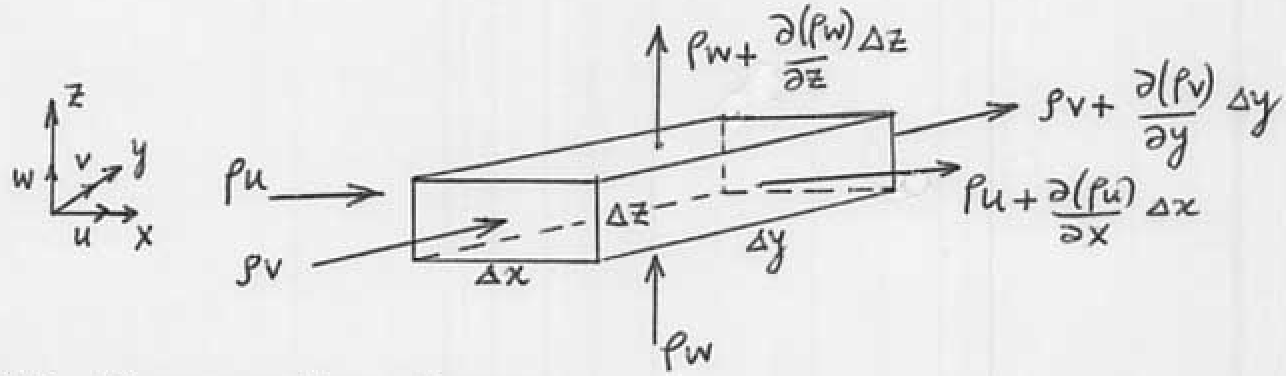
$$2\left(\frac{\partial v}{\partial y}\right) = \epsilon_{yy}, \tag{315}$$

$$2\left(\frac{\partial w}{\partial z}\right) = \epsilon_{zz}. \tag{316}$$

The concepts behind Equations 300, 302 and 316 must be carefully understood.

- We see that linear deformations are associated with normal strain rates. We would expect the normal strain rates to arise as a result of normal stresses.

Now, $\nabla \cdot \mathbf{V}$ represents the change of volume per unit volume of the fluid as the fluid element moves from one location to another in the flow field. If there is a volume change, then there has to be a density change. But, for an incompressible fluid, the density change in the flow field is zero. Therefore, **for an incompressible fluid, independent of whether the flow is steady or not, $\nabla \cdot \mathbf{V} = 0$** . Now we are ready to develop the conservation equations in differential forms. We shall concentrate on conservation of mass (continuity), conservation of linear momentum, and conservation of mechanical energy (isothermal flows).



6.2 Conservation of mass

The control volume is infinitesimally small and is about a point $P(x, y, z)$. u, v, w are velocity components of fluid velocity at P. With reference to the figure, remembering that flux denotes quantities per unit time and per unit area,

$$\text{Mass flux entering cv in x} - \text{direction} = \rho\, u \tag{317}$$

$$\text{Mass flux leaving cv in x} - \text{direction} = \left[\rho\, u + \frac{\partial(\rho\, u)}{\partial x}\Delta x\right] \tag{318}$$

$$\text{Area of either x face} = (\Delta y.\ \Delta z) \tag{319}$$

$$\text{Net mass leaving cv per unit time in x} - \text{direction} = \left[\frac{\partial\,(\rho\, u)}{\partial x}\ \Delta\ x\right]\ \Delta y\ \Delta z \tag{320}$$

Net mass leaving the cv per unit time from all directions is given by

$$\left[\frac{\partial\,(\rho\, u)}{\partial x}\ \Delta\ x\right]\ \Delta y\ \Delta z + \left[\frac{\partial\,(\rho\, v)}{\partial y}\ \Delta\ y\right]\ \Delta x\ \Delta z + \left[\frac{\partial\,(\rho\, w)}{\partial z}\ \Delta\ z\right]\ \Delta x\ \Delta y. \tag{321}$$

The net mass leaving the control volume must cause the mass in the cv to decrease. The rate of decrease of mass in the cv is

$$-\frac{\partial(\rho\, V_{cv})}{\partial t} = -V_{cv}\frac{\partial \rho}{\partial t} = -\Delta x\ \Delta y\ \Delta z \frac{\partial \rho}{\partial t}. \tag{322}$$

since the cv is non-deformable. From Equations 321 and 322,

$$\frac{\partial\,(\rho\, u)}{\partial x} + \frac{\partial\,(\rho\, v)}{\partial y} + \frac{\partial\,(\rho\, w)}{\partial z} = -\frac{\partial \rho}{\partial t} \tag{323}$$

$$\boldsymbol{\nabla} \cdot \rho\, \mathbf{V} = -\frac{\partial \rho}{\partial t} \tag{324}$$

$$\mathbf{V} \cdot (\boldsymbol{\nabla}\, \rho) + \rho\, (\boldsymbol{\nabla} \cdot \mathbf{V}) = -\frac{\partial \rho}{\partial t} \tag{325}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot (\boldsymbol{\nabla}\, \rho) + \rho\, (\boldsymbol{\nabla} \cdot \mathbf{V}) = 0 \tag{326}$$

Therefore, it follows that for an incompressible fluid, independent of whether the fluid flow is steady or not, $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$. We had already seen this result. In general, fluid flows must satisfy Equation 326. **The equation 326 will apply regardless of the choice of the coordinate system– rectangular cartesian, cylindrical, spherical etc.**

6.2.1 Steady, incompressible, planar (two-dimensional) flow and continuity equation

This is one of the simplest types of flow of practical importance. $\mathbf{V} = \mathbf{V}(x, y)$. Therefore, $\nabla \cdot \mathbf{V} = 0$ becomes,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{327}$$

Introduce a function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \tag{328}$$

The equation 328 shows that function ψ automatically satifies 327 and we can represent $u(x, y)$ and $v(x, y)$ by just one function $\psi(x, y)$. This function is is called the **Stream function**. Whenever the stream function formulation is used to describe a 2D flow problem, the continuity equation is automatically satisfied and we need not worry about it. However, the order of the accompanying equations will increase. We have to deal with that. A further advantage of the use of ψ is that ψ = constant denotes various streamlines in the flow for different values of the constant. How so? Recall the equation for a streamline in a 2D flow:

$$\frac{dy}{dx} = \frac{v}{u} \tag{329}$$

$$udy - vdx = 0 \tag{330}$$

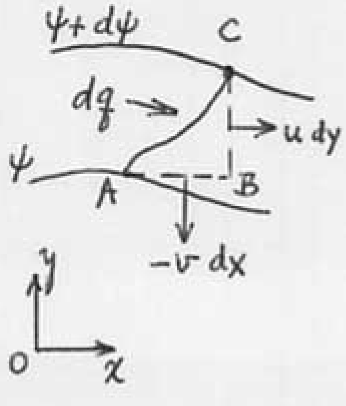
$$\frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx = 0 \tag{331}$$

$$\text{But, } \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx = d\psi, \text{ from chain rule} \tag{332}$$

$$\text{Along a stream line, } d\psi = 0 \tag{333}$$

$$\text{Therefore, along a stream line, } \psi = \text{constant} \tag{334}$$

The actual numerical value associated with a particular stream function and the streamline that it denotes is unimportant. But the difference between two stream functions in a flow is a measure of the quantity of fluid flow passing between the two streamlines per unit width perpendicular to the plane containing the streamlines. Flow never crosses a streamline. Looking at the figure, consider the flow bounded by streamlines with ψ and $\psi + d\psi$ for the stream functions. Let dq be the inflow crossing any arbitrary surface AC. This must equal the nèt out flow through surfaces AB and BC.

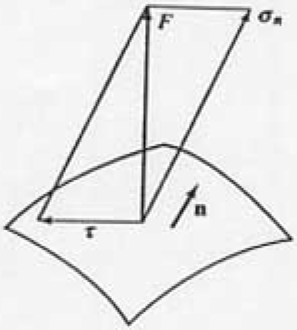


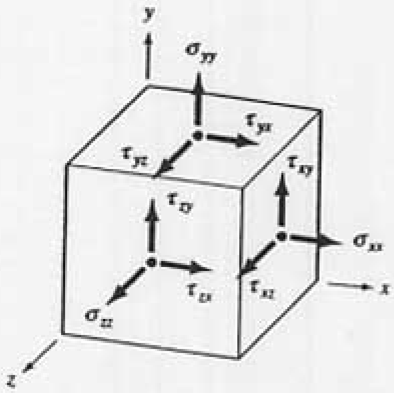
$$dq = udy - vdx \tag{335}$$

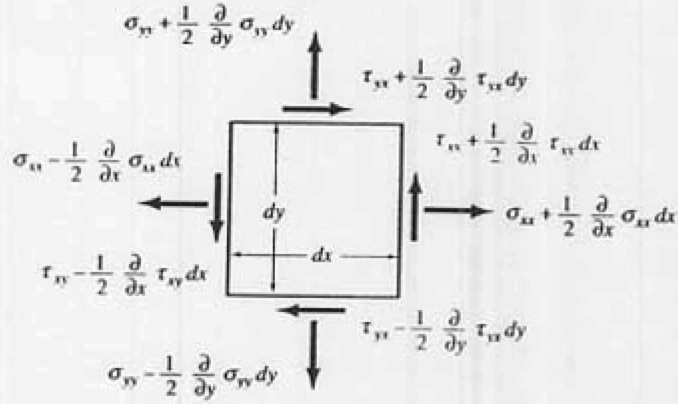
$$= \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx \tag{336}$$

$$= d\psi \tag{337}$$

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1 \tag{338}$$







6.3 Conservation of linear momentum for a viscous fluid: The Navier-Stokes equation

In a viscous fluid, evaluation of the surface forces are considerably more complicated. There are normal stresses similar to pressure, but they may not be the same in all directions.There are shear stresses whose directions are parallel to the surfaces on which they act. Look at the figures. Notation : τ_{yx} denotes stress acting in the x direction on a surface whose normal points in the y direction. This is therefore a shear stress. Similarly, σ_{xx} indicates a normal stress. Outward drawn normals indicate positive directions.

$$\sigma_{xx} = \lim_{dA_x \to 0} \frac{\mathbf{dF}_{sx}}{dA_x},\ \tau_{xy} = \lim_{dA_x \to 0} \frac{\mathbf{dF}_{sy}}{dA_x},\ \tau_{xz} = \lim_{dA_x \to 0} \frac{\mathbf{dF}_{sz}}{dA_x}, ... \tag{339}$$

To develop equations, we set the stresses at the center of the infinitesimal fluid element equal to $\sigma_{xx}, \tau_{xy}, \sigma_{yy}, ...$ and so on.

The net surface force in the x direction for a unit depth in the z-direction is:

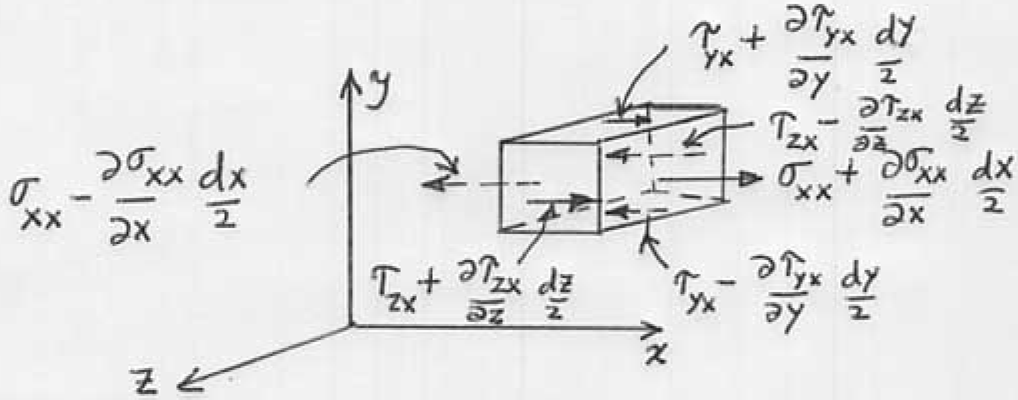
$$\frac{\partial(\sigma_{xx})}{\partial x}dydz + \frac{\partial(\tau_{yx})}{\partial y}dzdy, \tag{340}$$

and if three dimensions are taken into account, for the net surface force in the x direction we would obtain,

$$\frac{\partial(\sigma_{xx})}{\partial x}dydxdz + \frac{\partial(\tau_{yx})}{\partial y}dxdydz + \frac{\partial(\tau_{zx})}{\partial z}dzdxdy. \tag{341}$$

The net body force in the x direction is $\rho\, dxdydz\, \mathbf{f}_x$, where $\mathbf{f}_x$ is the net body force in the x direction per unit mass of fluid. The net total force in the x direction will give rise to acceleration in the x direction. We know that the acceleration component in the x direction is $\frac{Du}{Dt}$, noting that $\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ and $\frac{D}{Dt}$ is the material derivative with the observer flowing along with the fluid. So we have to equate the net force in the x direction to the acceleration imparted in the x direction to the fluid element of mass $\rho\, dxdydz$. Therefore, we obtain,

$$\rho\, dxdydz\, \frac{Du}{Dt} = \frac{\partial(\sigma_{xx})}{\partial x}dydxdz + \frac{\partial(\tau_{yx})}{\partial y}dxdydz + \frac{\partial(\tau_{zx})}{\partial z}dzdxdy + \rho\, dxdydz\, \mathbf{f}_x. \tag{342}$$



- More details on the devlopment of the net surface force in the x-direction,

$$\frac{\partial(\sigma_{xx})}{\partial x}dydxdz + \frac{\partial(\tau_{yx})}{\partial y}dxdydz + \frac{\partial(\tau_{zx})}{\partial z}dzdxdy. \tag{375}$$

To obtain the net surface force in the x-direction $\mathbf{dF_s}$, we must sum the forces in the x-direction. Thus, from the figure showing the stresses in the x-direction

$$\mathbf{dF}_{sx} = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz \tag{376}$$

$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial x}\frac{dy}{2}\right)dxdz \tag{377}$$

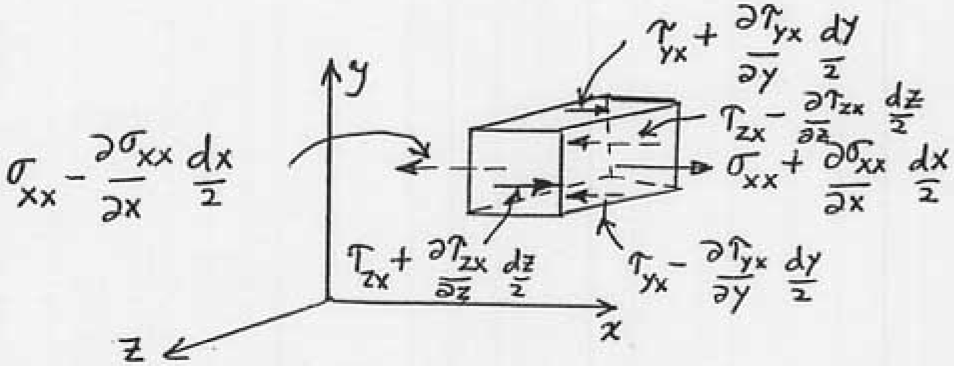
$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy \tag{378}$$

Upon simplifying,

$$\mathbf{dF}_{sx} = \frac{\partial(\sigma_{xx})}{\partial x}dxdydz + \frac{\partial(\tau_{yx})}{\partial y}dydxdz + \frac{\partial(\tau_{zx})}{\partial z}dzdxdy, \tag{379}$$

so that,

$$\mathbf{dF}_{bx} + \mathbf{dF}_{sx} = \left(\rho\, g_x + \frac{\partial(\sigma_{xx})}{\partial x} + \frac{\partial(\tau_{yx})}{\partial y} + \frac{\partial(\tau_{zx})}{\partial z}\right)dxdydz. \tag{380}$$



$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

In 342, We can divide throughout by $dxdydz$ to obtain the equation applicable for the x direction. Now, remember we are using the stress system in three dimensions shown in the determinant and we are also relying on the fact that the state of stress at a point can be described completely by specifying the stresses acting on three mutually perpendicular planes through the point (principal planes and principal axes). Thus, we can generalize 342 to three dimensions to obtain:

$$\rho\,\frac{Du}{Dt} = \frac{\partial(\sigma_{xx})}{\partial x} + \frac{\partial(\tau_{yx})}{\partial y} + \frac{\partial(\tau_{zx})}{\partial z} + \rho\,\mathbf{f}_x \tag{343}$$

$$\rho\,\frac{Dv}{Dt} = \frac{\partial(\tau_{xy})}{\partial x} + \frac{\partial(\sigma_{yy})}{\partial y} + \frac{\partial(\tau_{zy})}{\partial z} + \rho\,\mathbf{f}_y \tag{344}$$

$$\rho\,\frac{Dw}{Dt} = \frac{\partial(\tau_{xz})}{\partial x} + \frac{\partial(\tau_{yz})}{\partial y} + \frac{\partial(\sigma_{zz})}{\partial z} + \rho\,\mathbf{f}_z. \tag{345}$$

The above differential equations apply for any fluid motion satisfying the continuum assumption. However, we have to link the stresses to velocity and pressure fields. Now, of the nine stress elements, $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx}, \tau_{yz}, \tau_{zy}$ only six are independent, because of stress tensor symmetry. Thus, we can use the fact, $\tau_{yx} = \tau_{xy}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}$ to simplify things. From our discussions of rate of shear strain and rate of normal strain, we know that shear stresses must be proportional to rates of shear strain and normal stresses must be proportional to rates of normal strain.

In particular, for a Newtonian fluid, the stress-strain relationship on the basis of experimental observations and meaningful conjectures may be written,

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{346}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{347}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{348}$$

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\,\mu\,\nabla\cdot\mathrm{V} \tag{349}$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\,\mu\,\nabla\cdot\mathrm{V} \tag{350}$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\,\mu\,\nabla\cdot\mathrm{V} \tag{351}$$

where the coefficient of proportionality between the stress and the rate of strain is the dynamic viscosity of the fluid μ. In the above, p represents the "average pressure" and may be shown to be given by

$$p = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = p_{th} - \lambda \nabla \cdot \mathbf{V}, \tag{352}$$

where p_{th} is the thermodynamic pressure and is the one related to the density and temperature by the thermodynamic equation of state and λ is called the second coefficient viscosity. λ has the same dimensions as μ. In the development and application of fluid mechanics up to the present time, the second coefficient of viscosity plays much less of a role than the first. For incompressible fluids the term in λ disappears completely and for compressible fluids it is of significance mainly in a few specialized problems where very large velocity and temperature gradients may occur such as in the analysis of the shock wave structure or perhaps in the study of electric field effects on flames (I say 'perhaps' in the latter case because I don't know for sure if it is important but it could be) or in extreme situations invoving polyatomic gases. Anyway, few direct measurements of λ are available. Still, the distinction between the thermodynamic pressure and the average pressure must be appreciated and this is not always carefully made in books. With the expressions for the stresses introduced into them, the momentum equations become:

$$\rho\,\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) - \frac{2}{3}\frac{\partial}{\partial x}\left(\mu\, \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} + \rho\, f_x \tag{353}$$

$$\rho\,\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + 2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) - \frac{2}{3}\frac{\partial}{\partial y}\left(\mu\, \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} + \rho\, f_y \tag{354}$$

$$\rho\,\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) - \frac{2}{3}\frac{\partial}{\partial z}\left(\mu\, \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} + \rho\, f_z \tag{355}$$

These three equations are the three components of the **Navier-Stokes equation**. These equations are greatly simplified when applied to an incompressible flow ($\nabla \cdot \mathbf{V} = 0$) and with constant viscosity. Then these equations reduce to:

$$\rho\,\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho\, f_x \tag{356}$$

$$\rho\,\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho\, f_y \tag{357}$$

$$\rho\,\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho\, f_z \tag{358}$$

We can write these three component equations compactly as:

$$\rho\,\left\{\frac{\partial \mathbf{V}}{\partial t} + \left(\mathbf{V} \cdot \nabla\right)\mathbf{V}\right\} = -\nabla p + \mu \nabla^2 \mathbf{V} + \rho\, \mathbf{f} \tag{359}$$

or,

$$\rho\,\frac{D\mathbf{V}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{V} + \rho\, \mathbf{f} \tag{360}$$

- **Initial and Boundary conditions**

- The Navier-Stokes equation , 360,

$$\rho \left\{ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right\} = -\nabla p + \mu \nabla^2 \mathbf{V} + \rho \, \mathbf{f},$$

requires prescribed initial and boundary conditions. Initial conditions are dictated by the prevailing flow conditions at the start of the investigation or study. The boundary conditions must be carefully prescribed.

For flow over a solid surface, on the basis of experimental results, the tangential and normal components of the fluid are seen to be equal to the corresponding components of the surface itself. In other words, there is no relative motion or "slip" between the fluid and the solid. If the solid is at rest, then, for the fluid **V** is zero. If the surface is moving, then at the surface, the velocity of the fluid is equal to the velocity of the surface motion. However, when the free path of the molecules of the fluid becomes appreciable when compared to an important physical dimension of the body over which or through which the flow occurs (e.g., Rarefied gas flow at very high altitudes), the difference in tangential velocities between the fluid and the solid is not zero but is proportional to the surface shear stress. At a free surface or interface between two immiscible fluids, the shear stress must be continuous. If an appreciable surface tension exists at the interface, the normal component of the stress vector is discontinuous by an amount

$$\Delta p = \sigma \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \tag{361}$$

as discussed before.

The Navier-Stokes equation is difficult to solve, especially so when the flow is unsteady and/or fully three-dimensional. In a small number of cases, under some restrictive assumptions or because of the special nature of the physical problem which admits of simplifying assumptions, analytical solutions are possible. We will study a few of those cases soon. These days, many solutions for realistic formulations are developed employing extensive numerical schemes. With Super computers, we have made some progress, but fully 3-D Direct Numerical Simulation is still a very formidable task.

However, based on a thorough physical understanding, reasonable approximations may be introduced to develop simplified mathematical structures governing the momentum transport, and the resulting equations may be solved to gain a deep understanding of the flow field. Together with experimental observations, great progress may be made by such procedures. For example, following Prandtl, many flow fields may be divided into two regions, one close to the surface or boundary, the other involving the remainder of the flow. In the region close to the boundary, viscosity effects are deemed important, but the main portion of the fluid is regarded as inviscid (and perhaps, additionally,irrotational).

"Perfect" fluids do not actually exist aside possibly from liquid helium at temperatures near zero. Nevertheless, under certain conditions the behavior of an actual fluid away from a boundary approaches that of the "perfect" fluid. So we need to clearly understand an inviscid, irrotational, Incompressible flow and this will enable us treat a significant region of the flow field away from the boundary and later we may couple (or match) it with the treatment of the boundary layer. In the boundary layer itself, on the basis of the assumption of a physically thin layer, several approximations may be made in the process of the development of the viscous momentum equation which were not possible when we developed the N-S equation. We will learn about these soon. First, let us look at Inviscid, Irrotational, Incompressible flows.

6.4 Inviscid, Irrotational flows and the Velocity Potential

Recall that for a flow field with rotation,

$$2\,\omega = \nabla \times \mathbf{V} = \zeta, \tag{362}$$

and when,

$$\nabla \times \mathbf{V} = 0, \tag{363}$$

the flow is said to be **Irrotational**. Therefore, for an irrotational flow,

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}, \quad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \tag{364}$$

Now, from Vector calculus, for any scalar function $\phi(x, y, z, t)$ with continuous first and second derivatives,

$$\nabla \times \nabla\phi = 0. \tag{365}$$

Compare equations 363 and 365. It follows that for an irrotational flow we can always write,

$$\mathbf{V} = \nabla\phi, \tag{366}$$

Therefore,

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}. \tag{367}$$

For a two-dimensional irrotational flow, we need to consider,

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}. \tag{368}$$

Next consider a two-dimensional incompressible flow. Recall that we had introduced $\psi(x, y)$ as the stream function for such a flow and shown that it automatically satisfied the continuity equation $\nabla \cdot \mathbf{V} = 0$. The $\psi(x, y)$ as given by 328 was such that

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \tag{369}$$

Let such a two-dimensional, incompressible flow be irrotational as well. Then we have,

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \text{ and } -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \tag{370}$$

The equations given in 370 are known as the Cauchy-Riemann equations. These functions are "Harmonic". We can see that more explicitly by introducing the definitions of ϕ and ψ into the continuity equation and the irrotationality condition, respectively,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \text{ and } \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{371}$$

- We recognize the equations in 371 as forms of Laplace's equation. Any function ψ or ϕ that satisfies Laplace's equation represents a possible two-dimensional, incompressible, irrotational flow field.

We should remember that physically a fluid flow may follow a straight (rectilinear) path or an arbitrary curvilinear path. In either case, the flow may be, from our definition, "rotational" or "irrotational". Our definition of "rotationality" implies that the angle between two intersecting lines on a fluid element continues to change as a result of tangential stresses. Tangential stresses arise as a result of viscosity. Therefore, viscous fluid motion is "rotational" whether the motion is straight or not. On the otherhand inviscid ($\mu = 0$) flow may be "irrotational" or "rotational". The rotationality may be introduced into an inviscid flow by external work interaction (forced vortex or rigid body rotation) or heat transfer. A free vortex (swirling motion of the water as it drains from a bath tub) is irrotational except at the origin or center. Thus for an irrotational flow all the three: friction, external work input and heat transfer should be absent. This is the criterion.

- **Orthogonality of the lines of constant ψ and ϕ**

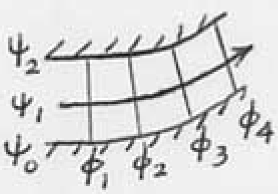
In a 2-D flow, along a given streamline, the stream function ψ is a constant, and therefore, $d\psi = 0$. This means,

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0 \tag{372}$$

$$-vdx + udy = 0 \tag{373}$$

$$\frac{dy}{dx}\Big|_{\psi=c} = \frac{v}{u} \text{ (slope of streamline)} \tag{374}$$

On the otherhand, along a line of constant ϕ, $d\phi = 0$. This means,



$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0 \tag{375}$$

$$udx + vdy = 0 \tag{376}$$

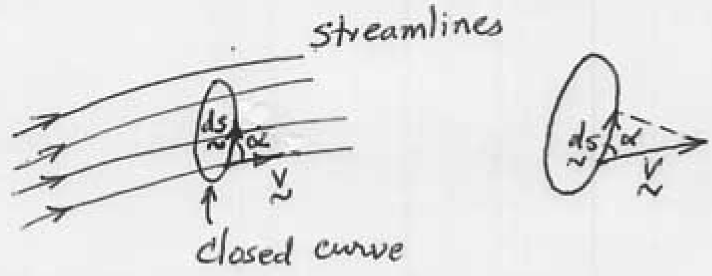
$$\frac{dy}{dx}\Big|_{\phi=c} = -\frac{u}{v} \text{ (slope of potential flow line)} \tag{377}$$

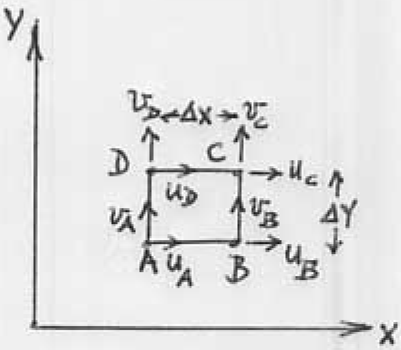
Therefore, from, 374 and 377, lines of constant ψ and ϕ are orthogonal at any given point. These lines form an orthogonal network. When such a network is formed, from the spacing of the ψ and ϕ lines velocities can be computed, and the pressure may then be determined from Bernoulli's equation. Since there is no flow through any of the streamlines, such as ψ_1 or ψ_2, any one of them could also be considered to be a possible solid boundary. This feature helps in the graphical analyses of irrotational flow fields.

6.4.1 Representation in Polar coordinates

Recall that in cylindrical coordinates, $\nabla = \mathbf{i_r}\frac{\partial}{\partial r} + \mathbf{i_\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{i_z}\frac{\partial}{\partial z}$. The stream function, $\psi(r, \theta, t)$ is defined such that,

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r}, \quad \text{and,} \quad V_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \tag{378}$$



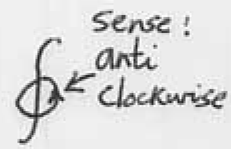


6.4.2 Concept of Circulation and Irrotational Flow

Circulation in fluid mechanics is defined as the line integral of the tangential velocity component of a fluid flow about a closed curve that is fixed in the flow. For example, in a 2-D flow field, it may be noted that each streamline will intersect a closed curve fixed in the flow at some angle α, and thus the component of velocity along the closed curve at the point of intersection is $|\mathbf{V}|\cos\alpha = V\cos\alpha$ = tangential component of velocity. Then, we define an element of circulation $d\Gamma$ and circulation Γ by:

$$d\Gamma = \mathbf{V}\cdot\mathbf{dS} \;=\; V\cos\alpha \qquad (379)$$

$$\Gamma \;=\; \oint \mathbf{V}\cdot\mathbf{dS} \qquad (380)$$



$\oint$ indicates that the integral is to be taken once around the closed curve and the direction convention is counterclockwise. Clearly, calculation of circulation around an arbitrary curve in a flow field is generally a tedious step-by-step integration but for circles and squares it is easy. For example, consider the closed curve which is a square as shown. To calculate, proceed from A counterclockwise.

$$\begin{aligned} d\Gamma &= (\text{Mean velocity along AB})\,\Delta x + (\text{Mean velocity along BC})\,\Delta y \\ &- (\text{Mean velocity along CD})\,\Delta x - (\text{Mean velocity along DA})\,\Delta y \qquad (381) \\ &= \left(\frac{u_A + u_B}{2}\right)\Delta x + \left(\frac{v_B + v_C}{2}\right)\Delta y \\ &- \left(\frac{u_C + u_D}{2}\right)\Delta x - \left(\frac{v_A + v_D}{2}\right)\Delta y \qquad (382) \end{aligned}$$

Now, $u_A = u$, $u_B = u + \frac{\partial u}{\partial x}\Delta x$, $u_C = u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y$, $u_D = u + \frac{\partial u}{\partial y}\Delta y$, $v_A = v$, $v_B = v + \frac{\partial v}{\partial x}\Delta x$, $v_C = v + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y$, $v_D = v + \frac{\partial v}{\partial y}\Delta y$. Therefore,

$$d\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x\,\Delta y \qquad (383)$$

in which $\Delta x\Delta y$ is the enclosed boundary. Therefore,

$$\frac{d\Gamma}{\Delta x\,\Delta y} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (384)$$

Recall, $\underset{\sim}{\zeta} \equiv \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \equiv \underset{\sim}{i}\,\zeta_x + \underset{\sim}{j}\,\zeta_y + \underset{\sim}{k}\,\zeta_z$

Here, we recognize

$$\frac{d\Gamma}{\Delta x\,\Delta y} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \zeta_z = \text{z} - \text{component of Vorticity } \zeta. \tag{385}$$

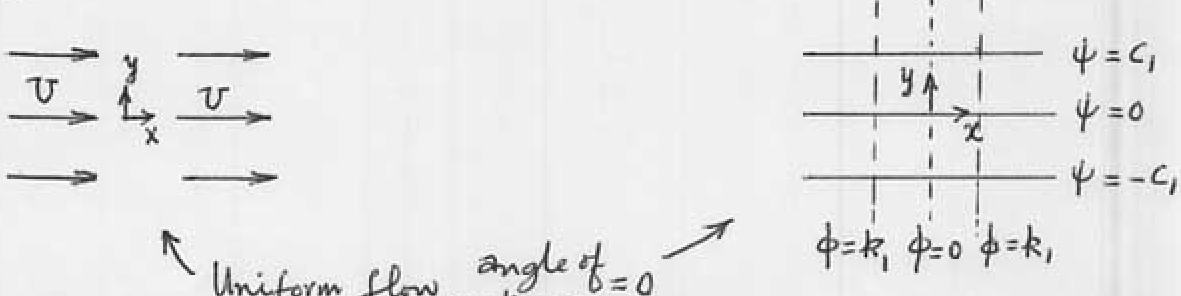
Therefore, we conclude that vorticity component denotes the differential circulation per unit area enclosed. So, Vorticity is some measure of the rotational aspects of the fluid elements as they move through the field of flow. If Vorticity is absent, the flow is Irrotational. Where the flow is irrotational, the circulation is zero as well. Now, an entire flow field need not be either rotational or irrotational. Actually flowfields may possess separate zones of both rotational and irrotational flows. This last feature is an important feature enabling the development of boundary layer theory.

6.4.3 Examples of Potential Flows

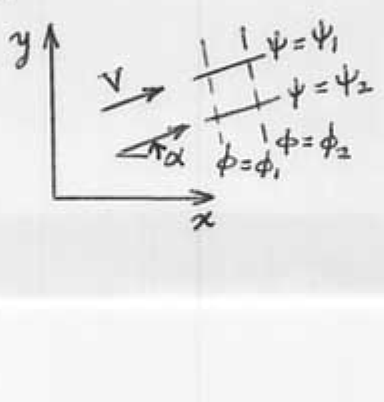
- **A Uniform flow**

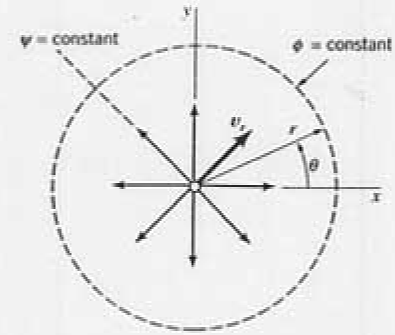
In a uniform flow field, the magnitude and direction of the velocity vector are constant throughout the field. Consider a uniform flow parallel to the x axis in the positive x direction. This flow satisfies the continuity equation and the irrotationality. Here, $u = U$, and $v = 0$. Therefore, $\psi = Uy$ and $\phi = Ux$.

For a uniform flow of constant magnitude V, inclined at an angle α to the x axis, $\psi = (V\cos\alpha)y - (V\sin\alpha)x$ and $\phi = (V\cos\alpha)x + (V\sin\alpha)y$. $\Gamma = 0$ around any closed curve.



Uniform flow, angle of attack = α ⇒





- **Source and Sink flows**

A simple source is a flow pattern in the xy plane in which the flow is radially outward from the z axis (origin) and symmetrical in all directions. If the flow moves radially inward, the pattern is called a sink. Sources and sinks can be used to approximate some aspects of real flows at points away from the origin. Due to the radial geometry, we shall use cylindrical polar coordinates to analyze source and sink flows. The angle θ is measured positive in the anticlockwise direction. We have $x = r\cos\theta$, $y = r\sin\theta$, and $x^2 + y^2 = r^2$. First consider the source.

Let m be the volume flow rate, emanating from the line (z-axis) per unit depth. This is called the **strength** of the source. At any radius, r, from the source, since the flow is purely radial, the tangential velocity v_θ is zero; and, the radial velocity, v_r is the volume flow rate per unit depth, m, divided by the flow area per unit depth, $(2\pi r)(1) = 2\pi r$. Therefore, for a source,

$$v_r = \frac{m}{2\pi r} \tag{386}$$

Then,

$$v_r = \frac{\partial\phi}{\partial r} = \frac{m}{2\pi r}, \text{ and } v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0 \tag{387}$$

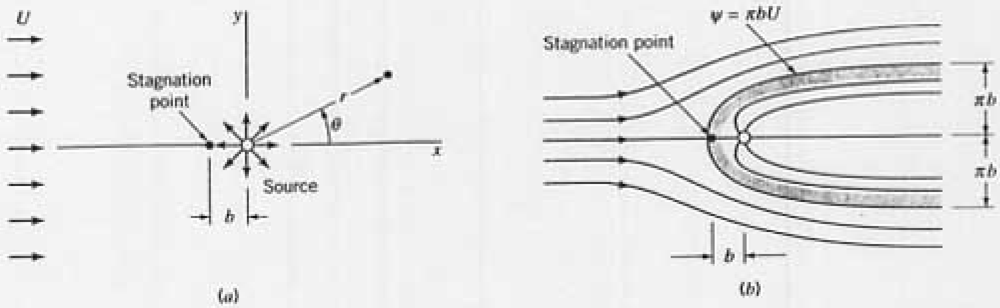
Therefore,

$$\phi = \frac{m}{2\pi}\ln r \tag{388}$$

The equipotential lines (ϕ = constant) are therefore concentric circles centered at the origin. If m is positive, the flow is radially outward, and we have a source flow. If m is negative, the flow is toward the origin, and we have a sink flow. The stream function for the source may be obtained from

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{m}{2\pi r}, \text{ and } \frac{\partial\psi}{\partial r} = 0, \Rightarrow \psi = \frac{m}{2\pi}\theta. \tag{389}$$

The streamlines (ψ = constant) are radial lines. At the origin, $r = 0$, the velocity becomes infinite, and this is physically meaningless. The reason for this is that we have assumed that a physical fluid flow with volume flow rate m to emanate from a line of unit depth along the $z - axis$. We consider the line at the origin as a mathematical **singularity** in the flow field. A function ϕ or ψ is not continuous at a singular point and the circulation along a path crossing the origin cannot be evaluated. However, for a path surrounding or circumventing the origin, the circulation is zero since $v_\theta = 0$, and the flow is irrotational everywhere but the origin.



6.4.4 Superposition of a uniform stream and a source; Half-body problem

Recall that both ϕ and ψ satisfy Laplace's equation for flow that is both incompressible and irrotational. Since Laplace'e equation is a linear, homogeneous partial differential equation, solutions may be superposed (added together) to develop more complex and interesting patterns of flow.

Consider the superposition of a source and a uniform flow as shown in the figure.

$$\psi = \psi_{uf} + \psi_{sou} \tag{390}$$

$$= Ur\sin\theta + \frac{m}{2\pi}\theta, \tag{391}$$

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r \tag{392}$$

At some point on the negative x axis, say, at a distance b from the source, the velocity due to the source will just cancel that due to the uniform flow and a stagnation point will be created. At distance b on the negative x axis, the v_r due to source is $v_r = \frac{m}{2\pi r} = \frac{m}{2\pi b}$. Therefore, at the location b away,

$$U = \frac{m}{2\pi b} \tag{393}$$

$$\text{and, } b = \frac{m}{2\pi U} \tag{394}$$

The coordinates for the stagnation point are seen to be $r = b,\ \theta = \pi$. Therefore, $\psi_{\text{stagnation}} = \frac{m}{2}$. Therefore, from 394, at the stagnation point,

$$\frac{m}{2} = \pi bU, \text{ or } \frac{m}{2\pi} = bU. \tag{395}$$

From 391 and 395, the equation of the streamline passing through the stagnation point $r = b, \theta = \pi$ is,

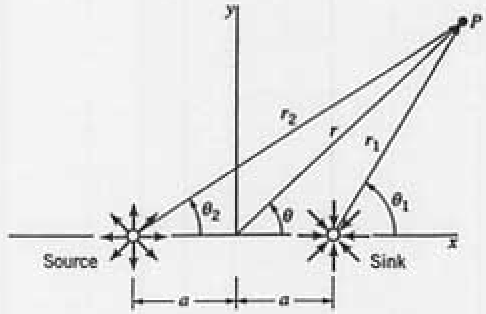
$$\psi_{\text{stagnation}} = \frac{m}{2} = \pi bU = Ur\sin\theta + \frac{m}{2\pi}\theta = Ur\sin\theta + bU\theta, \tag{396}$$

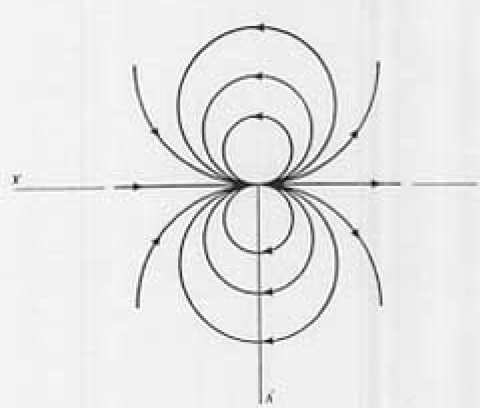
or, the relation between r and θ for this line is,

$$\pi bU = Ur\sin\theta + bU\theta \tag{397}$$

$$r = \frac{b(\pi - \theta)}{\sin\theta}, 0 \le \theta \le 2\pi. \tag{398}$$

We can plot this streamline (see, figure). Since there is no flow across a streamline, we can regard this streamline as a solid boundary. Thus the combination of a uniform flow and a source can be used to describe the potential flow over a streamlined body placed in the flow. The body is open at the downstream end and thus is called a Half-body. We can plot other streamlines in the flow field by giving different values to ψ and plotting the resulting equation.





- **Doublet**

Similar to a dipole in electro-statics, the fluid mechanical doublet is a combination of a source and a sink of equal strength, and spaced a small distance apart. In the Figure, the source at $(-a, 0)$ and the sink at $(a, 0)$, each of strength m, are located on the x-axis on either side of the origin. Physically, the flow leaving the source terminates on the sink. A point $P(x, y)$ in the flow field has polar coordinates (r, θ), and is at a distance r_2 from the source and r_1 from the sink. Thus, the velocity potential for both at P is,

$$\phi = -\frac{m}{2\pi}\ln r_1 + \frac{m}{2\pi}\ln r_2 = \frac{m}{2\pi}\ln\frac{r_2}{r_1} \tag{399}$$

The two dimensional doublet is defined as the limiting case as a source and sink of equal strength approach each other, $(a \to 0)$, such that the product of the strength and the distance between them remains a constant. So, our objective now is to evaluate ϕ as $(a \to 0)$. Look at the figure. From geometry,

$$r_1^2 = r^2 + a^2 - 2ar\cos\theta = r^2\left[1 + \left(\frac{a}{r}\right)^2 - 2\frac{a}{r}\cos\theta\right] \tag{400}$$

$$r_2^2 = r^2 + a^2 + 2ar\cos\theta = r^2\left[1 + \left(\frac{a}{r}\right)^2 + 2\frac{a}{r}\cos\theta\right] \tag{401}$$

$$\frac{r_2^2}{r_1^2} = \frac{1 + \frac{2ra\cos\theta}{(r^2+a^2)}}{1 - \frac{2ra\cos\theta}{(r^2+a^2)}} \tag{402}$$

$$\text{Let, } \alpha = \frac{2\ r\ a\ \cos\theta}{(r^2 + a^2)}, \text{ note, } \alpha \ll 1 \tag{403}$$

$$\frac{r_2^2}{r_1^2} = (1 + \alpha)(1 - \alpha)^{-1} \tag{404}$$

$$= 1 + 2\alpha, \text{ neglecting higher order terms} \tag{405}$$

$$\text{Therefore, } \ln\frac{r_2}{r_1} = \frac{1}{2}\ln(1 + 2\alpha) \tag{406}$$

Since the distance between the source and the sink, $a \to 0$, we have, $2\,\alpha \ll 1$, and

$$\lim_{a\to 0} \ln \frac{r_2}{r_1} = \frac{1}{2}\left(2\alpha - \frac{4\alpha^2}{2} + \frac{8\alpha^3}{3} - ...\right) \tag{407}$$

$$\lim_{a\to 0} \ln \frac{r_2}{r_1} = \alpha \tag{408}$$

$$\lim_{a\to 0} \ln \frac{r_2}{r_1} = \frac{2\;r\;a\;\cos\theta}{(r^2+a^2)} \tag{409}$$

Therefore, the velocity potential for both at P as $a \to 0$, is,

$$\phi = \frac{m}{2\pi} \ln \frac{r_2}{r_1} = \frac{m}{2\pi} \frac{2\;r\;a\;\cos\theta}{(r^2+a^2)} \tag{410}$$

But $a \to 0$, Therefore, can neglect a^2. Thus, for a doublet,

$$\phi = \frac{m\;a}{\pi} \frac{\cos\theta}{r} \tag{411}$$

The quantity,

$$K = \frac{m\;a}{\pi} \tag{412}$$

is called the strength of the doublet. The value of K is maintained finite by increasing m to infinite value in the limit a is reduced to zero. Next, by relating ϕ and ψ, we can easily show that the stream function for the doublet is given by

$$\psi = -\frac{K \sin\theta}{r} \tag{413}$$

For a doublet, the streamlines and equipotential lines are both circles.

6.4.5 Bernoulli's equation for a steady, incompressible, inviscid, irrotational flow

Steady, Inviscid flow. The Navier- Stokes equation is, 360,

$$\rho\,\frac{D\mathbf{V}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{V} + \rho\,\mathbf{f} \tag{414}$$

reduces to

$$(\mathbf{V}\cdot\nabla)\,\mathbf{V} = -\frac{\nabla p}{\rho} + \mathbf{f} \tag{415}$$

Consider writing the body force due to gravity as $\mathbf{f} = -g\mathbf{k} = -g\nabla\mathbf{z}$. Then, 415 may be written as

$$(\mathbf{V}\cdot\nabla)\,\mathbf{V} = -\frac{\nabla p}{\rho} - g\nabla\mathbf{z} \tag{416}$$

Recognize this as 416 and the Euler's equation that we developed before using streamline coordinates. From Vector algebra,

$$(\mathbf{V}\cdot\nabla)\,\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) - \mathbf{V}\times(\nabla\times\mathbf{V}) \tag{417}$$

For an irrotational flow, $\mathbf{V} \times (\nabla \times \mathbf{V}) = 0$. Therefore, the vector identity becomes,

$$(\mathbf{V} \cdot \nabla)\,\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) = \frac{1}{2}\nabla(V^2) \tag{418}$$

From, 416 and 418,

$$\frac{1}{2}\nabla(V^2) = -\frac{1}{\rho}\nabla p - g\nabla \mathbf{z} \tag{419}$$

Now, during the interval, dt, a fluid particle moves from the vector position $\mathbf{r}$ to the position $\mathbf{r} + \mathbf{dr}$; the displacement $\mathbf{dr}$ is an arbitrary infinitesimal displacement in any direction. Take the dot product of $\mathbf{dr} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$, with each of the terms in 419. Then

$$\frac{1}{2}\nabla(V^2) \cdot \mathbf{dr} = -\frac{1}{\rho}\nabla p \cdot \mathbf{dr} - g\nabla \mathbf{z} \cdot \mathbf{dr} \tag{420}$$

This gives,

$$-\frac{dp}{\rho} - g dz = \frac{1}{2}d(V^2) \tag{421}$$

Since **dr** was an arbitrary displacement, then, for a steady, incompressible, inviscid flow that is also irrotational, 421 is valid between any two points in the flow field.

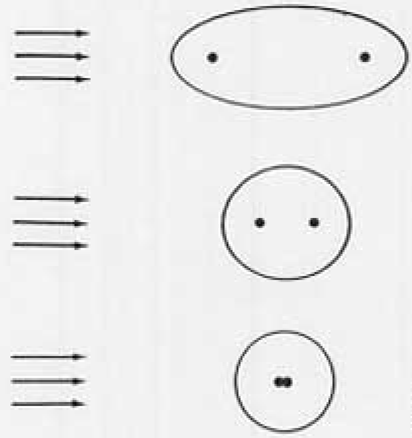
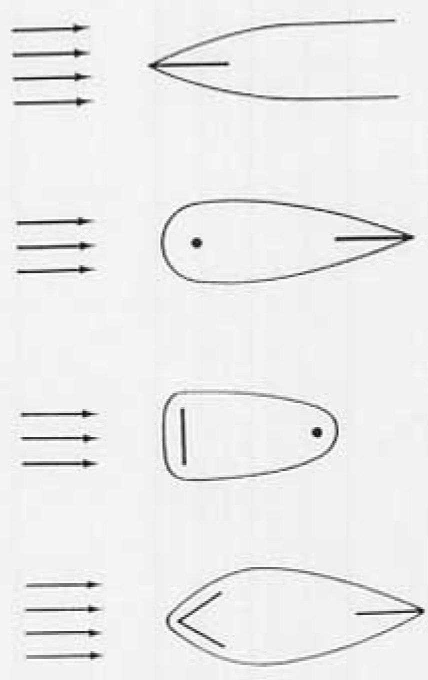


Figure Rankine bodies with varying source–sink distance.

By combining a number of sources and sinks with uniform flow, we can obtain a variety of bodies with different shapes. However, in order to obtain a closed body, the sum of the source strengths must equal the sum of the sink strengths. Thus far we have discussed only two-dimensional point sources and sinks. For the purpose of generating different bodies, we could also use sources and sinks distributed over a surface or even over a volume. When combined with uniform flow, a further variation in the body shapes obtainable results. Examples of different body shapes are shown in Figure where several two-dimensional source and sink distributions have been combined with a uniform stream.



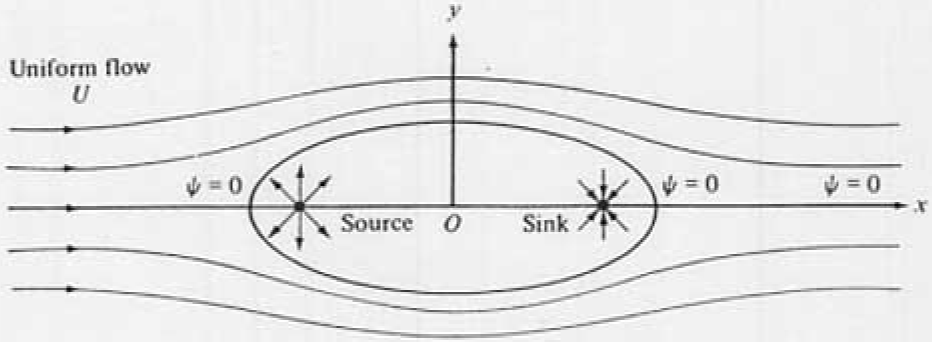
Pointed half body obtained from line source of linearly increasing strength parallel to stream

Streamline body obtained from point source and line sink of linearly decreasing strength

Flat-nosed body obtained from constant line source normal to stream and point sink

Slender-nosed streamline body obtained from wedge line source and sink of linearly decreasing strength

Figure Examples of body shapes obtainable.



- **Superposition of a source-sink pair and a uniform flow**

From the figure and the previous development, the potential function and the stream function for this combination are:

$$\phi = Ur\cos\theta - \frac{m}{2\pi}(\ln r_1 - \ln r_2) \quad (422)$$

$$\psi = Ur\sin\theta - \frac{m}{2\pi}(\theta_1 - \theta_2). \quad (423)$$

In terms of rectangular cartesian coordinates,

$$\phi = Ux + m\left[\ln\sqrt{(x+a)^2+y^2} - \ln\sqrt{(x-a)^2-y^2}\right] \quad (424)$$

$$\psi = Uy - m\, tan^{-1}\frac{2ay}{x^2+y^2-a^2} \quad (425)$$

The velocity components for the flow field are:

$$u = \frac{\partial\phi}{\partial x} = U + m\left[\frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right] \quad (426)$$

$$v = \frac{\partial\phi}{\partial y} = m\left[\frac{y}{(x+a)^2+y^2} - \frac{y}{(x-a)^2+y^2}\right] \quad (427)$$

To locate the stagnation points in the flow field, we have to look for points where u and v vanish. We see that $v = 0$ at $y = 0$, that is along the x axis.Thus we have to next look for $u = 0$ for $y = 0$. That is,

$$U + m\left[\frac{x+a}{(x+a)^2} - \frac{x-a}{(x-a)^2}\right] = 0 \quad (428)$$

$$U = \frac{2ma}{x^2-a^2} \quad (429)$$

$$x = \pm\sqrt{a^2 + \frac{2ma}{U}} \quad (430)$$

which states that the distance of the stagnation points (there are two in this flow case) from the coordinate origin is a function of the source (sink) strength, magnitude of the uniform flow velocity, and the distance between the source and the sink. The value of the stream function at the stagnation points, since $y = 0$, is

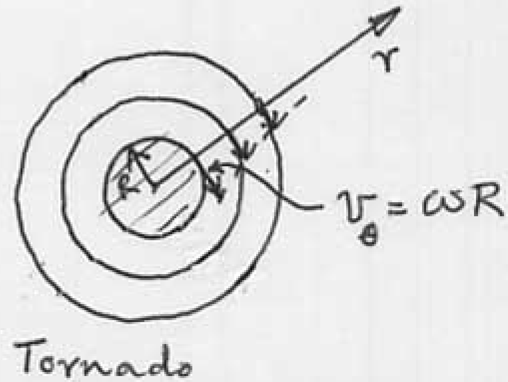
$$\psi_{\text{stagnation}} = 0 - m\, tan^{-1} \frac{0}{x^2 - a^2} = 0 \tag{431}$$

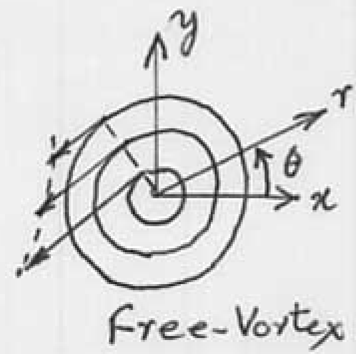
The equation of the body streamline shown in the figure therefore becomes

$$0 = Uy - m\, tan^{-1} \frac{2ay}{x^2 + y^2 - a^2} \tag{432}$$

or,

$$\tan \frac{Uy}{m} = \frac{2ay}{x^2 + y^2 - a^2}. \tag{433}$$





- **Combined Vortex Flow: Tornado Problem**

For a solid body rotation, we know that, $v_\theta = \omega r$ and this represents a forced vortex because external work or heat transfer or a combination is present. The vorticity is 2ω. To understand a free vortex, consider the swirling motion of water as it drains from a bath tub. Far from the drain, the water swirls around the drain in almost circular streamlines, with very little motion toward the drain. The swirling velocity increases as the flow moves toward the center. The circumferential velocity is such that $rv_\theta = \text{Constant} = C$. The origin is a singular point. The streamlines are concentric circles. The vorticity of a free vortex is

$$\zeta_z = 2\omega_z = \frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r}\right) - \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta}\right) \quad (434)$$

$$= \frac{1}{r}\left(\frac{\partial r \frac{C}{r}}{\partial r}\right) - \frac{1}{r}0 \quad (435)$$

$$= 0. \quad (436)$$

The flow field in a stationary tornado can be represented by a solid body rotational flow, " forced vortex", in the core (eye), while the flowfield outside the eye is an irrotational "free" vortex. Let R be the radius of the eye. Thus, for a tornado,

$$\text{In the core, } r \le R, v_\theta = \omega r \quad (437)$$

$$\text{Outside the core, } v_\theta = \frac{C}{r} \quad (438)$$

$$\text{At }, r = R, v_\theta = \omega r = \frac{C}{r} \Rightarrow C = \omega R^2 \quad (439)$$

$$\text{Therefore, for } r \ge R, v_\theta = \frac{\omega R^2}{r} \quad (440)$$

The maximum wind velocity occurs at the edge of the core, $\omega = \frac{v_\theta}{R}$. Outside the core, the flow is potential and we can apply Bernoulli's equation. In the core, we have to use Euler's equation. Minimum pressure occurs at $r = 0$, the center of the tornado, and will be negative. For a maximum wind velocity of $50m/s$, ($\sim 112mph$), the pressure at the center will be $\sim -3.063kPa$. Recall $1atm = 101.3kPa$.