$$\mathcal{L}_x \mathcal{L}_y - \mathcal{L}_y \mathcal{L}_x = -\hbar^2 \Big(\big(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\big)\big(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\big) - \big(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\big)\big(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\big)\Big)$$

$$= -\hbar^2 \Big(y\frac{\partial}{\partial z}\big(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\big) - z\frac{\partial}{\partial y}\big(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\big) - z\frac{\partial}{\partial x}\big(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\big) + x\frac{\partial}{\partial z}\big(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\big)\Big)$$

$$= -\hbar^2 \Big(y\big(z\frac{\partial^2}{\partial z\partial x} + \frac{\partial}{\partial x} - x\frac{\partial^2}{\partial z^2}\big) - z^2\frac{\partial^2}{\partial y\partial x} + zx\frac{\partial^2}{\partial y\partial z} - zy\frac{\partial^2}{\partial x\partial z} + z^2\frac{\partial^2}{\partial x\partial y} + xy\frac{\partial^2}{\partial z^2} - x\big(z\frac{\partial^2}{\partial z\partial y} + \frac{\partial}{\partial y}\big)\Big)$$

$$= -\hbar^2\big(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\big) = i\hbar\,\mathcal{L}_z$$

ص ٩٨

$$[\mathcal{L}_z , \nabla^2]\psi = \mathcal{L}_z \nabla^2\psi - \nabla^2 \mathcal{L}_z \psi$$

$$= \frac{\hbar}{i}\Big(\big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\big)\nabla^2\psi - \nabla^2\big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\big)\psi\Big)$$

$$= \frac{\hbar}{i}\Big(\big(x\frac{\partial}{\partial y}\nabla^2 - y\frac{\partial}{\partial x}\nabla^2\big)\psi - \nabla^2\big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\big)\psi\Big)$$

But

$$\nabla^2\big(x\frac{\partial}{\partial y}\big)\psi = \big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\big)\big(x\frac{\partial\psi}{\partial y}\big)$$

$$= \frac{\partial^2}{\partial x^2}\big(x\frac{\partial\psi}{\partial y}\big) + \frac{\partial^2}{\partial y^2}\big(x\frac{\partial\psi}{\partial y}\big) + \frac{\partial^2}{\partial z^2}\big(x\frac{\partial\psi}{\partial y}\big)$$

$$= \frac{\partial}{\partial x}\big(x\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial\psi}{\partial y}\big) + x\frac{\partial^3\psi}{\partial y^3} + x\frac{\partial^3\psi}{\partial z^2\partial y}$$

$$= x\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x\partial y} + x\frac{\partial^3\psi}{\partial y^3} + x\frac{\partial^3\psi}{\partial z^2\partial y}$$

$$= x\frac{\partial}{\partial y}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial z^2}\big)\psi + 2\frac{\partial^2\psi}{\partial x\partial y}$$

$$= x\frac{\partial}{\partial y}\nabla^2\psi + 2\frac{\partial^2\psi}{\partial x\partial y}$$

وبالمثل (أي نبدل y بـ x وبالعكس)

$$\nabla^2\big(y\frac{\partial}{\partial x}\big)\psi = y\frac{\partial}{\partial x}\nabla^2\psi + 2\frac{\partial^2\psi}{\partial y\partial x}$$

$$\Rightarrow [\mathcal{L}_z , \nabla^2]\psi = \frac{\hbar}{i}\Big(x\frac{\partial}{\partial y}\nabla^2 - y\frac{\partial}{\partial x}\nabla^2 - x\frac{\partial}{\partial y}\nabla^2 - 2\frac{\partial^2}{\partial x\partial y} + y\frac{\partial}{\partial x}\nabla^2 + 2\frac{\partial^2}{\ldots}\Big)\psi = 0$$