المحاضرة السابعة

or :

Let $\{I_k\}$ be a sequence of open intervals, $A \subset \bigcup_k I_k \Rightarrow A+y \subset \bigcup_k (I_k+y)$

$$\therefore m^*(A+y) \leq \sum_k L(I_k+y) = \sum L(I_k)$$
$$\leq m^*(A) \quad ①$$

To prove the reverse

Let $A' = A+y$, $y' = -y$

$$\Rightarrow m^*(A'+y') = m^*(A+y-y) \leq m^*(A+y)$$
$$\Rightarrow m^*(A) < m^*(A+y) \quad ②$$
$$\therefore m^*(A) = m^*(A+y), \quad y \in \mathbb{R}, \ A \subset R.$$

Proposition :

For any set $A \subset R$ and any $\varepsilon > 0$, there is an open set V such that

$$A \subset V, \quad m^*(V) \leq m^*(A) + \varepsilon.$$

Proof.

$\because m^*(A) = \inf \{ \sum_k L(I_k) : I_k \in I, \ A \subseteq \bigcup I_k, \ I_k \text{ - open \& bounded interval} \}$

$$\therefore \sum_k L(I_k) - \frac{\varepsilon}{2} < m^*(A)$$

If $I_k = (a_k, b_k)$, Let $J_k = \left(a_k - \frac{\varepsilon}{2^{k+1}}, b_k + \frac{\varepsilon}{2^{k+1}}\right)$,

$$V = \bigcup_k J_k$$
$$\Rightarrow A \subset \bigcup_k J_k,$$
$$\Rightarrow m^*(V) \leq \sum_k L(J_k) = \sum_k L(I_k) + \frac{\varepsilon}{2}$$
$$\leq m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$\leq m^*(A) + \varepsilon$$
$$\therefore m^*(V) \leq m^*(A) + \varepsilon, \quad A \subset V, \ V \text{ - is an open set.}$$

note :

$$m^*(V) \leq \sum_k m^*(J_k) = \sum_k m^*\left(a - \frac{\varepsilon}{2^{k+1}}, b_k + \frac{\varepsilon}{2^{k+1}}\right) = \sum_k L\left(a - \frac{\varepsilon}{2^{k+1}}, b_k + \frac{\varepsilon}{2^{k+1}}\right)$$
$$= \sum_k L(I_k) + \sum_k \frac{\varepsilon}{2^k} = \sum_k L(I_k) + \frac{\varepsilon}{2}$$

Proposition:

For any $a \in R$, the set (a,∞) is measurable.

proof.

Let $A \subseteq \mathbb{R}$, we show that

$$m^*(A) \geqslant m^*(A \cap (a,\infty)) + m^*(A \cap (a,\infty)^c)$$
$$= m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty, a])$$

Assume that $m^*(A) < \infty$. From definition of m^*

$\forall\ \varepsilon > 0\ \exists\ \{I_n\}$ - open, bounded interval such that

$$m^*(A) + \varepsilon > \sum_n L(I_n).$$

Let $J_n = I_n \cap (a,\infty)$ and $J_n' = I_n \cap (-\infty, a]$, then

$$J_n \cap J_n' = I_n \cap (a,\infty) \cap I_n \cap (-\infty, a]$$
$$= I_n \cap ((a,\infty) \cap (-\infty, a])$$
$$= I_n \cap \varphi = \varphi$$

$$J_n \cup J_n' = (I_n \cap (a,\infty)) \cup (I_n \cap (-\infty, a])$$
$$= I_n$$

Therefore,

$$L(I_n) = L(J_n \cup J_n') = L(J_n) + L(J_n') = m^*(J_n) + m^*(J_n')$$

since

$A \cap (a,\infty) \subset \bigcup_n J_n$ and $A \cap (-\infty, a] \subset \bigcup_n J_n'$

then

$$m^*(A \cap (a,\infty)) \leq m^*(\bigcup_n J_n) \leq \sum_n m^*(J_n) = \sum_n L(J_n)$$

and

$$m^*(A \cap (-\infty, a]) \leq m^*(\bigcup_n J_n') \leq \sum_n m^*(J_n') = \sum_n L(J_n')$$

Therefore

$$m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty, a])$$
$$\leq \sum_n [L(J_n) + L(J_n')] = \sum_n L(I_n)$$
$$\leq m^*(A) + \varepsilon$$

since ε is arbitrary

$$\therefore m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty, a]) \leq m^*(A)$$

Corollary :-

[1] Every interval in $\mathbb{R}$ is measurable.

proof.

① $(a,\infty) \subset \mathbb{R}$, (a,∞) is measurable. $\forall a \in \mathbb{R}$.

② $[-\infty, a] \subset \mathbb{R}$, $(a,\infty)^c = (-\infty, a]$ so $(-\infty, a]$ is measurable $\forall a \in \mathbb{R}$.

③ $[a,\infty) \subset \mathbb{R}$, $[a,\infty) = \{a\} \cup (a,\infty)$

$\because$ $m^*(\{a\}) = 0 \implies \{a\}$ is measurable.

$\because$ $\{a\}, (a,\infty) \subset \mathbb{R}$, $\{a\}$ and (a,∞) is measurable

$\therefore$ $[a,\infty)$ is measurable.

④ $(-\infty, a) = [a,\infty)^c \implies (-\infty, a)$ is measurable.

⑤ $(a,b) = (-\infty, b) \cap (a,\infty)$, $a < b$, $a, b \in \mathbb{R}$.

$\because$ $(-\infty, b)$ and (a,∞) are both measurable.

$\because$ Ω is collection of all measurable subsets of $\mathbb{R}$.

and Ω is an algebra

$\therefore$ (a,b) is also measurable.

⑥ $[a,b) = \{a\}$, $(a,b] = (a,b) \cup \{b\}$ and $[a,b] = \{a\} \cup (a,b) \cup \{b\}$.

$\therefore$ The sets $[a,b)$, $(a,b]$ and $[a,b]$ are all measurable.

[2] Every open set in $\mathbb{R}$ is measurable.

proof.

$\because$ Every open set in $\mathbb{R}$ is a countable union of open intervals.

$\because$ Every open intervals is measurable

and Ω is an algebra

$\therefore$ Every open set is measurable.

$\therefore$

[3] Every closed set in $\mathbb{R}$ is measurable

proof

Let F - closed set $\implies F^c$ is an open set

$\because$ Every open set is measurable.

$\therefore$ F is measurable.

4. Every set that is a Countable intersection of open sets in $\mathbb{R}$ is measurable.

Proof.

Let $U = \bigcap_{i=1}^{\infty} V_i$ where V_i - open set in $\mathbb{R}$.

$\because$ V_i - measurable

$\because$ V_i^c - Closed set $\Rightarrow$ V_i^c - measurable.

$\because$ Ω is an σ-algebra

$\therefore$ $U = \bigcap_{i=1}^{\infty} V_i = \left(\bigcup_{i=1}^{\infty} V_i^c\right)^c$

$\therefore$ U is measurable.

5. Every set that is a Countable union of closed sets in $\mathbb{R}$ is measurable.

Proof.

Let $F = \bigcup_{i=1}^{\infty} F_i$, where F_i is closed in $\mathbb{R}$

$\Rightarrow$ F_i^c is open set $\Rightarrow$ F_i^c is measurable.

$\Rightarrow$ $F = \bigcup_{i=1}^{\infty} F_i = \left(\bigcap F_i^c\right)^c$

$\Rightarrow$ F is measurable.

Definition.

1. A set that is a union of a Countable Collection of Closed sets is Called a F_σ - set. $F = \bigcup_{i=1}^{\infty} F_i \rightarrow F_i$ - closed

2. A set that is an intersection of a Countable Collection of open sets is Called a G_δ - set.

$U = \bigcap_{i=1}^{\infty} V_i$, V_i - open set

Remark.

1. Every F_σ - set and every G_δ - set is measurable.

2. The smallest σ-algebra Containing all the open sets Called the Borel σ-algebra.