المحاضرة السادسة

since $A\cap(E_1\cup E_2)\subseteq A$. By replacing A by $A\cap(E_1\cup E_2)$

$$\therefore \mu^*[A\cap(E_1\cup E_2)] = \mu^*[(A\cap(E_1\cup E_2))\cap E_1] + \mu^*[(A\cap(E_1\cup E_2))\cap E_1^c]$$

$$= \mu^*[((A\cap E_1)\cup(A\cap E_2))\cap E_1] + \mu^*[((A\cap E_1)\cup(A\cap E_2))\cap E_1^c]$$

$(A\cap E_1\cap E_1)\cup(A\cap E_2\cap E_1)$ $\quad (A\cap E_1\cap E_1^c)\cup(A\cap E_2\cap E_1^c)$

$(A\cap E_1)\cup A$ $\quad A \quad \cup$

$$= \mu^*(A\cap E_1) + \mu^*(A\cap E_2)$$

Lemma 3.

Let X be a set and μ^* an outer measure on X. suppose that (E_n) be a sequence of mutually disjoint sets in Ω (the collection of all μ^*-measurable subsets of X). Then

$$\mu^*(A\cap\bigcup_{i=1}^{n}E_i) = \sum_{j=1}^{n}\mu^*(A\cap E_i), \quad \forall A\subseteq X,\ n\geq 1.$$

proof.

We prove this Lemma by induction.

at $n=1$

$$\mu^*(A\cap E_1) = \sum_{j=1}^{1}\mu^*(A\cap E_j) = \mu^*(A\cap E_1) \qquad \forall A\subseteq X,\ E_1\in\Omega$$

at $n=2$

$$\mu^*(A\cap\bigcup_{j=1}^{2}E_j) = \mu^*(A\cap(E_1\cup E_2)) = \mu^*((A\cap E_1)\cup(A\cap E_2))$$
$$= \mu^*(A\cap E_1) + \mu^*(A\cap E_2) \quad \text{From Lemma 2.}$$

we assume that

$$\mu^*(A\cap\bigcup_{j=1}^{n-1}E_j) = \sum_{j=1}^{n-1}\mu^*(A\cap E_j)$$

since E_n is μ^*-measurable,

$$\mu^*(A\cap\bigcup_{j=1}^{n}E_j) = \mu^*[(A\cap\bigcup_{j=1}^{n}E_j)\cap E_n] + \mu^*[(A\cap\bigcup_{j=1}^{n}E_j)\cap E_n^c]$$

$(A\cap E_n)\bigcup_{j=1}(E_i\cap E_n)$ $\qquad E_n\cap E_n^c=\varphi$

$$= \mu^*(A\cap E_n) + \mu^*(A\cap(\bigcup_{j=1}^{n-1}E_j))$$

$$= \mu^*(A\cap E_n) + \sum_{j=1}^{n-1}\mu^*(A\cap E_j)$$
$$= \sum_{j=1}^{n}\mu^*(A\cap E_j)$$

$$\mu((A\cap\bigcup_{j=1}^{n}E_i)\cap E_n^c) = \mu((A\cap E_n^c)\bigcup_{j=1}^{n}(E_j\cap E_n^c)) = \mu(A\cap\bigcup_{j=1}^{n-1}E_j)$$

Remark. μ^* is finitely additive on Ω.

By taking $A = X$, we obtain

$$\mu^*\left[A \cap \bigcup_{j=1}^{n} E_j\right] = \mu^*\left[X \cap \bigcup_{j=1}^{n} E_j\right] = \mu^*\left[\bigcup_{j=1}^{n} E_j\right] =$$
$$= \sum_{j=1}^{n} \mu^*(A \cap E_j) = \sum_{j=1}^{n} \mu^*(X \cap E_j)$$
$$= \sum_{j=1}^{n} \mu^*(E_j).$$

Theorem. (Carathéodory's Theorem).

If μ^* is an outer measure on a set X, then the following statements hold:

a) The family

$$\Omega = \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\},$$

is a σ-algebra on X.

b) $\mu = \mu^*|_\Omega$ is a measure on (X, Ω).

c) μ is a complete measure. In fact,

$\forall\ E \subseteq X$, that satisfies $\mu^*(E) = 0$ is μ^*-measurable.

proof.

(a) ① $\because \mu^*(\varphi) = 0$, then φ is μ^*-measurable.

and from definition X is μ^*-measurable.

$\therefore X, \varphi \in \Omega$.

② $\because E$ is μ^*-measurable, we compute that

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall\ A \subseteq X$$
$$= \mu^*(A \cap E^c) + \mu^*(A \cap (E^c)^c)$$
$$= \mu^*(A \cap E^c) + \mu^*(A \cap E)$$

Hence E^c is μ-measurable, so $E^c \in \Omega$.

③ we will show that Ω is closed under finite unions?

If $E, F \in \Omega$, then $E \cup F \in \Omega$ From Lemma 1.

④ Now, we show that Ω is closed under countable disjoint unions.

Let (E_n) be a pairwise disjoint sequence of sets in Ω and define

$$S = \bigcup_{k=1}^{\infty} E_k \quad \text{and} \quad S_n = \bigcup_{k=1}^{n} E_k, \quad n \in \mathbb{N}.$$

Since $S_n \in \Omega$, since Ω is closed under finite unions.

$$\mu^*(A) = \mu^*(A \cap S_n) + \mu^*(A \cap S_n^c)$$

since $S_n \subset S \Rightarrow S^c \subset S_n^c$ and μ^* is monotone

$$\therefore \mu^*(A) \geq \mu^*(A \cap S_n) + \mu^*(A \cap S^c)$$
$$\geq \sum_{j=1}^{n} \mu^*(A \cap E_j) + \mu^*(A \cap S^c)$$

Since this is true for every natural number n,

$$\therefore \mu^*(A) \geq \sum_{j=1}^{\infty} \mu^*(A \cap E_j) + \mu^*(A \cap S^c)$$
$$\geq \mu^*(A \cap S) + \mu^*(A \cap S^c)$$

(since μ^* is countably subadditive)

Therefore $S = \bigcup_{j=1}^{\infty} E_j \in \Omega$.

⑤ Ω is closed under finite intersection

$$E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$$
$$\because E_1, E_2 \in \Omega \Rightarrow E_1^c, E_2^c \in \Omega$$
$$\Rightarrow E_1^c \cup E_2^c \in \Omega$$
$$\Rightarrow (E_1^c \cup E_2^c)^c \in \Omega$$
$$\therefore E_1 \cap E_2 \in \Omega.$$

Therefore, Ω is an σ-algebra.

(b) Let $\mu^* : \Omega \to [0, \infty) \Rightarrow \mu^*$ is a measure on Ω.

① $\mu^*(\phi) = 0$, By definition of the outer measure.

② Let (E_n) be a pairwise disjoint sequence of sets in Ω.

From (4)

$$\because \mu^*(A) \geq \sum_{j=1}^{\infty} \mu^*(A \cap E_j) + \mu^*\left[A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right]$$
$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \geq \sum_{j=1}^{\infty} \mu^*\left(\bigcup_{i=1}^{\infty} E_i \cap E_j\right) + \mu^*\left[\bigcup_{i=1}^{\infty} E_i \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right]$$
$$\geq \sum_{j=1}^{\infty} \mu^*(E_j) + \mu^*(\phi)$$
$$\geq \sum_{j=1}^{\infty} \mu^*(E_j)$$

Since μ^* is countable subadditive, we have

$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu^*(E_j)$$

we conclude that

$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu^*(E_j)$$

Thus, μ^* is a measure on Ω.

Lebesgue measure of $\mathbb{R}$.

We use the approach discussed in the previous section to construct the Lebesgue measure on $\mathbb{R}$. The Lebesgue measure will lead to the definition of the Lebesgue integral in the next chapter. Let $\mathcal{I}$ be a collection of open intervals in $\mathbb{R}$. For $I \in \mathcal{I}$, let $L(I)$ denote the length of I.

Theorem.

For $A \subseteq \mathbb{R}$, define $m^* : P(\mathbb{R}) \longrightarrow [0, \infty)$ by

$$m^*(A) = \inf \left\{ \sum_n L(I_n) : I_n \in \mathcal{I}, \text{ for each } n \text{ and } A \subseteq \bigcup_n I_n \right\}$$

Then m^* is an outer measure on $\mathbb{R}$.

proof.

① Since $\varphi \subseteq (a, a)$, for any real number a, then

$$0 \leq m^*(\varphi) \leq a - a = 0 \implies m^*(\varphi) = 0.$$

② Let A and B be subsets of $\mathbb{R}$, $A \subseteq B$. If $B \subseteq \bigcup_n I_n$ (where (I_n) is a sequence of open interval in $\mathbb{R}$)

$$\therefore A \subseteq B \subseteq \bigcup_n I_n \implies \sum_n L(I_n) \in \cancel{X}(B) \subset \cancel{X}(A)$$

$$\implies m^*(A) = \inf(X_A) \leq \inf(X_B) \leq m^*(B).$$

$X(A) = \{ \sum_{n=1}^{\infty} L(I_n) : A \subseteq \bigcup I_n \}$ $\quad X(B) = \{ \sum_{n=1}^{\infty} L(I_n) : B \subset \bigcup_n I_n \}$.

③ Let (A_n) be a sequence of subsets of $\mathbb{R}$.

① If $m^*(A_n) = \infty$ for some n,

$$\therefore m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$$

suppose $m^*(A_n) < \infty$ for each n. Then $\forall n$ and given $\varepsilon > 0$ there is a sequence $(I_{nk})_k$ of open interval such that

$$A_n \subset \bigcup_{k=1}^{\infty} I_{nk} \quad \text{and}$$

$$\sum_{k=1}^{\infty} l(I_{nk}) < m^*(A_n) + \frac{\varepsilon}{2^n}$$

Since $\bigcup_n A_n \subseteq \bigcup_n \bigcup_{k=1}^{\infty} I_{n,k}$

$$\therefore m^*(\bigcup_n A_n) \leq \sum_n \sum_k l(I_{nk}) < \sum_n (m^*(A_n) + \frac{\varepsilon}{2^n})$$
$$= \sum_n m^*(A_n) + \varepsilon \sum_n \frac{1}{2^n}$$

Since ε is arbitrary, we have that $m^*(\bigcup_n A_n) \leq \sum_n m^*(A_n)$.

Definition:-

A set $E \subseteq \mathbb{R}$ is said to be Lebesgue-measurable (or measurable), if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

for every $A \subseteq \mathbb{R}$.

Remark.

* Denote by Ω, the set of all Lebesgue-measurable subsets of $\mathbb{R}$, then Ω is a σ-algebra.
* $m = m^*|_\Omega$ is a measure on Ω. This measure is called the Lebesgue measure on $\mathbb{R}$.

Properties of the Lebesgue outer measure.

Proposition 1.

The outer measure of an interval I is its length.

i.e $m^*(I) = L(I)$.

Proof.

① Let $I = [a,b]$, & bounded. then $[a,b] \subset (a-\varepsilon, b+\varepsilon) \quad \forall\, \varepsilon > 0$

$$\therefore m^*([a,b]) \leq L(a-\varepsilon, b+\varepsilon) = b-a+2\varepsilon.$$

since ε is arbitrary

$$\therefore m^*(I) \leq b-a = L(I).$$

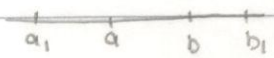
suppose that $I \subset \bigcup_{k=1}^{\infty} I_k$, I_k is open for each $k \in \mathbb{N}$.

since I is compact,

$\therefore$ there is a finite subcollection $\{I_k\}_{k=1}^{m}$ such that

$$I \subset \bigcup_{k=1}^{m} I_k.$$

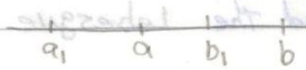
since $a \in I \implies \exists\, (a_1, b_1)$ in $\{I_k\}_{k=1}^{m}$ such that $a_1 < a < b_1$.



a) If $b \leq b_1 \implies a_1 < a < b$ $\quad (b \in (a_1, b_1))$

$$\therefore \sum_{k=1}^{n} L(I_k) = \sum_{k=1}^{m} l(a_1, b_1) \geq b_1 - a_1$$
$$\geq b-a$$

$$\therefore \sum_{k=1}^{n} L(I_k) \geq b-a$$

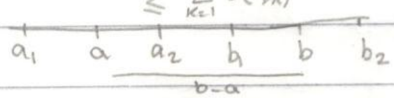


b) If $b_1 < b \implies b_1 \in (a,b)$ & $b_1 \notin (a_1, b_1)$

$\therefore b_1 \in [a,b] \implies \exists\, (a_2, b_2)$ in $\{I_k\}_{k=1}^{m}$ such that

$$b_1 \in (a_2, b_2)$$

$L(I) = b-a \leq b_2 - a_1 = (b_2 - a_2) + (b_1 - a_1)$
$\leq \sum_{k=1} L(I_k)$



b') If $b \leq b_2$, then we are done

b'') if $a < b_2 < b$, then $\exists\, (a_3, b_3)$ in $\{I_k\}_{k=1}^{m}$ such that

$$b_2 \in (a_3, b_3).$$

Continuing, we obtain a sequence of intervals

$(a_1, b_1), (a_2, b_2), \ldots, (a_5, b_5)$ from $\{I_k\}_{k=1}^{m}$ such that

$$a_i < b_{i-1} < b_i \quad \forall\, i = 2, 3, \ldots, 5.$$

since $\{I_k\}_{k=1}^{m}$ is finite, $s \leq m$ steps with $b < b_s$.

$$\sum_{j=1}^{\infty} L(I_j) \geq \sum_{j=1}^{m} L(I_j) \geq \sum_{j=1}^{s} L(a_j, b_j)$$

$$= (b_s - a_s) + (b_{s-1} - a_{s-1}) + \cdots + (b_1 - a_1)$$

$$= b_s - (\overset{-ve}{a_s - b_{s-1}}) - (\overset{-ve}{a_{s-1} - b_{s-2}}) - \cdots - (\overset{-ve}{a_2 - b_1}) - a_1$$

$$> b_s - a_1 > b - a$$

$$\therefore \; m^*(I) \geq b - a \implies m^*(I) = L(I).$$

② Let I be an interval of the form (a,b), $(a,b]$, $[a,b)$.
Then for each $0 < \varepsilon < b-a$ with

$$(a + \tfrac{\varepsilon}{2}, b - \tfrac{\varepsilon}{2}) \subset I \subset (a - \tfrac{\varepsilon}{2}, b + \tfrac{\varepsilon}{2})$$

Hence

$$m^*((a + \tfrac{\varepsilon}{2}, b - \tfrac{\varepsilon}{2})) \leq m^*(I) \leq m^*((a - \tfrac{\varepsilon}{2}, b + \tfrac{\varepsilon}{2}))$$

$$\iff L((a + \tfrac{\varepsilon}{2}, b - \tfrac{\varepsilon}{2})) \leq m^*(I) \leq L((a - \tfrac{\varepsilon}{2}, b + \tfrac{\varepsilon}{2}))$$

$$\iff b - a - \varepsilon \leq m^*(I) \leq b - a + \varepsilon$$

since ε is arbitrary $\implies m^*(I) = b - a = L(I)$.

③ suppose that I is an unbounded interval. $\implies$
$\exists$ a close interval $J \subset I$ such that $L(J) = K$, K - real.

Hence,

$$m^*(I) \geq m^*(J) = L(J) = K$$

Thus

$$m^*(I) = \infty, \qquad L(I) = \infty$$

Proposition:

For each $x \in \mathbb{R}$, $m^*(\{x\}) = 0$.

Proof.

$\{x\} \subset (x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$. Thus

$$0 \le m^*(\{x\}) \le m^*(x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}) = L(x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2})$$
$$\le x + \tfrac{\varepsilon}{2} - x + \tfrac{\varepsilon}{2} = \varepsilon$$

Since ε is arbitrary, we have $m^*(\{x\}) = 0$.

② Corollary.

Every Countable subset of $\mathbb{R}$ has outer measure Zero.

proof.

Let A be a Countable subset of $\mathbb{R}$. Then $A = \bigcup_n \{a_n\}$, therefore.

$$0 \le m^*(A) = m^*(\bigcup_n \{a_n\}) \le \sum_n m^*(\{a_n\})$$
$$\le 0$$

Thus

$$m^*(A) = 0.$$

Proposition.

The Lebesgue outer measure is translation-invariant. that is

$$m^*(A+y) = m^*(A) \quad \forall \; A \subset \mathbb{R}, \; y \in \mathbb{R}$$

proof.

Note that:

① $\{I_K\}_{K=1}^{\infty}$ is a Cover for A iff $\{I_{K+y}\}_{K=1}^{\infty}$ is a Cover of $A+y$

② I_K is open and bound. intervals $\Rightarrow$ I_{K+y} is also open and bounded

③ $L(I_K) = L(I_{K+y})$.

$\therefore$ $m^*(A) = \inf \{ \sum L(I_K) : A \subseteq \bigcup I_K, \; I_K \text{ is open \& bounded interval} \}$

$= \inf \{ \sum L(I_{K+y}) : A+y \subseteq \bigcup_K I_{K+y}, \; I_{K+y} \text{ is open \& bounded} \}$

$= m^*(A+y) \quad \forall \; A \subset \mathbb{R}$.