Example

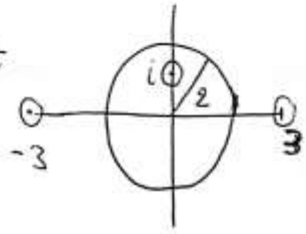
Evaluate the integral $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$

where c is the circle $|z|=2$ taken in the positive sense.

Solution

Since the fn. $f(z)=\frac{z}{(9-z^2)}$ is analytic within and on C, we can apply the Cauchy integral formula to it when $z=-i$

$$F^{(n)}(\xi)=\frac{n!}{2\pi i}\int_C \frac{F(z)}{(z-\xi)^{n+1}}\,dz$$



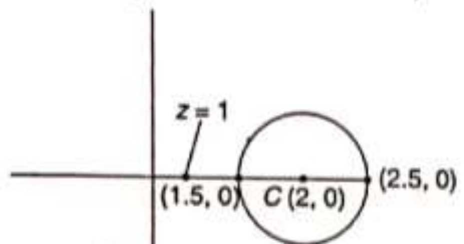
$$2\pi i\, f(-i) = 2\pi i\left(\frac{-i}{9+1}\right) = \frac{\pi}{5}$$

1. Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)}$ C is $|z-2| = \frac{1}{2}$.

Solution: $\frac{z}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$,

$$[B = 2,\ A = -1]$$

$$\frac{z}{(z-1)(z-2)} = \frac{2}{(z-2)} - \frac{1}{(z-1)}$$



[$z = 2$ lies inside and $z = 1$ lies outside]

$$\int_C \frac{z\,dz}{z-1} = 0$$

$$\int_C \frac{z\,dz}{(z-1)(z-2)} = \int_C \frac{2z\,dz}{z-2}$$

$$\left[\begin{array}{l} f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)},\ a = 2,\ f(z) = 2z \\ f(2) = 4 \end{array}\right]$$

$$\int_C \frac{z\,dz}{(z-1)(z-2)} = 4\pi i.$$

- - z

3. Evaluate $\int_C \frac{ze^z dz}{(z+2)^3}$ where C is $|z| = 3$. **(JNTU 2006 Nov.)**

Solution: $z = -2$ lies inside $|z| = 3$

According to Cauchy's integral formula

$$f''(a) = \frac{1}{\pi i}\int_C \frac{f(z)dz}{(z-a)^3},\ [f(z) = ze^z\ a = -2]$$

$$f'(z) = ze^z + e^z,$$

$$f''(z) = ze^z + 2e^z$$

$$f''(-2) = 0$$

$\therefore$ $$\int_C \frac{ze^z dz}{(z+2)^3} = 0.$$

4. Evaluate $\int_C \left[\frac{e^z}{z^3} + \frac{z^4}{(z+i)^2}\right]dz$, where $C : |z| = 2$.

(JNTU 2004 May, 2004 Nov., 2006 Nov., 2008 April/May)

Solution: $$\int_C \left[\frac{e^z}{z^3} + \frac{z^4}{(z+i)^2}\right]dz = \int_C \frac{e^z dz}{z^3} + \int_C \frac{z^4 dz}{(z+i)^2}$$

$$= I_1 + I_2 \text{ (say)}$$

I_1

According to Cauchy's integral formula

$$f''(a) = \frac{1}{\pi i}\int_C \frac{f(z)dz}{(z-a)^3}\ [f(z) = e^z,\ a = 0]$$

O lies inside C

$$f''(z) = e^z \text{ and } f''(0) = 1$$

$\therefore$ $$\int_C \frac{e^z dz}{z^3} = \pi i$$

I_2

According to Cauchy's integral formula

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}\ [a = -i,\ z = -i \text{ lies inside } C]$$

$$f(z) = z^4,\ f'(z) = 4z^3 \text{ and } f'(-i) = 4i$$

$\therefore$ $$4i = \frac{1}{2\pi i}\int_C \frac{z^4 dz}{(z+i)^2}$$

$\therefore$ $$\int_C \frac{z^4 dz}{(z+i)^2} = -8\pi$$

$\therefore$ $$\int_C \left[\frac{e^z}{z^3} + \frac{z^4}{(z+i)^2}\right] dz = \pi i - 8\pi.$$

6. Evaluate $\int_C \frac{(z^3 - \sin 3z)dz}{\left(z - \frac{\pi}{2}\right)^3}$ with $C: |z| = 2$, using Cauchy's integral formula. **(JNTU 2005 April, 2005 Nov., 2007 Feb., 2008 Nov.)**

Solution: According to Cauchy's integral formula

$$f''(a) = \frac{1}{\pi i}\int \frac{f(z)dz}{(z-a)^3} \quad \left[f(z) = z^3 - \sin 3z \text{ and } a = \frac{\pi}{2}\right]$$

$\frac{\pi}{2} < 2$, $z = \frac{\pi}{2}$ lies inside $C: |z| = 2$

$$f''(z) = 6z + 9\sin 3z$$

$$f''\left(\frac{\pi}{2}\right) = 3\pi - 9$$

$\therefore$ $$\int_C \frac{f(z)dz}{(z-a)^3} = \pi i(3\pi - 9).$$

8. Evaluate $\int_C \frac{dz}{e^z (z-1)^3}$ where C is $|z| = 2$ using Cauchy's integral theorem.

(JNTU 2005 April, 2006 Aug., 2008 Nov.)

Solution:

$$\int_C \frac{dz}{e^z (z-1)^3} = \int_C \frac{e^{-z} dz}{(z-1)^3}$$

$$z = 1 \text{ lies inside } C: |z| = 2$$

$$f(z) = e^{-z}$$

According to Cauchy's integral theorem

$$\frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)} = f(a), \ [a = 1]$$

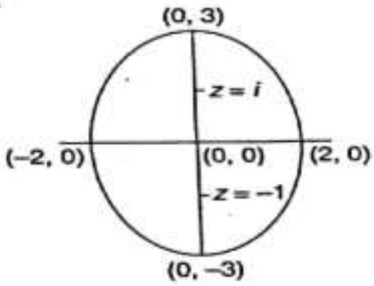
$$f''(a) = \frac{1}{\pi i} \int \frac{f(z)dz}{(z-a)^3}$$

$$f''(z) = e^{-z}, f''(1) = e^{-1}$$

$$\therefore \qquad \int_C \frac{e^{-z} dz}{(z-1)^3} = \frac{\pi i}{e}.$$

12. Using Cauchy's integral formula evaluate $\int_C \frac{z^4 dz}{(z+1)(z-i)^2}$ where C is ellipse and $9x^2 + 4y^2 = 36$.

(JNTU 2004 Nov.)



Solution: $\int_C \frac{z^4 dz}{(z+1)(z-i)^2}$

$$= \int_C \frac{z^4 dz}{(1+i)^2 (z+1)} - \int_C \frac{z^4 dz}{(1+i)^2 (z-i)} + \frac{1}{(1+i)} \int_C \frac{z^4 dz}{(z-i)^2}$$

Splitting into partial fractions

$$z = -1 \text{ and } z = i \text{ lie inside } 9x^2 + 4y^2 = 36$$

According to Cauchy's integral theorem

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)}$$

$$\frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^2} = f'(a)$$

$$f(z) = z^4, \ a = -1, \ f(-1) = 1, \ a = i, \ f(i) = 1$$

$$f'(z) = 4z^3 \text{ and } f'(i) = -4i$$

$$\therefore \quad \int_C \frac{z^4 dz}{(z+1)(z-i)^2} = \frac{1}{(1+i)^2} \cdot 2\pi i - \frac{1}{(1+i)^2} \cdot 2\pi i + \frac{1}{(1+i)} 2\pi i(-4i)$$

$$= \frac{8\pi}{} = 4\pi(1-i).$$

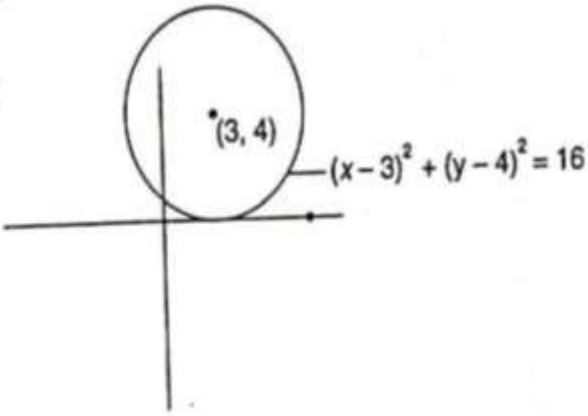
17. Evaluate using Cauchy's theorem $\int_C \frac{z\,dz}{(z^2-6z+25)^2}$ where

$C: |\, z-(3+4i)\,| = 4.$ **(JNTU 2005 April)**

Solution: According to Cauchy's integral formula

$$2\pi i\, f'(a) = \int_C \frac{f(z)dz}{(z-a)^2}$$

$z = 3 + 4i$ is inside C



$$f(z) = \frac{z}{(z-3+4i)^2}$$

$$f'(z) = \frac{-3+4i-z}{(z-3+4i)^3}$$

$$f'(3+4i) = -\frac{3i}{256}$$

$$\int_C \frac{z\,dz}{(z^2-6z+25)^2} = 2\pi i\left(-\frac{3i}{256}\right) = \frac{3\pi}{128}.$$

19. **Evaluate** $\int_C \frac{e^z dz}{(z^2+\pi^2)^2}$ **where** C **is** $|\, z\,| = 4$.

(JNTU 2004 April, 2006 Nov., 2008 April/May, 2008 Nov.)

Solution: $$\frac{1}{(z^2+\pi^2)^2} = \frac{A}{(z+\pi i)} + \frac{B}{(z+\pi i)^2} + \frac{C}{(z-\pi i)} + \frac{D}{(z-\pi i)^2}$$

$$\left[A = \frac{i}{4\pi^3}, B = -\frac{1}{4\pi^2}, C = -\frac{i}{4\pi^3} \text{ and } D = -\frac{1}{4\pi^2}\right]$$

$\therefore \quad A(z-\pi i)^2(z+\pi i) + B(z-\pi i)^2 + C(z+\pi i)^2(z-\pi i) + D(z+\pi i)^2 = 1$

Put

$$z = \pi i,\ D(2\pi i)^2 = 1 \text{ and } D = -\frac{1}{4\pi^2}$$

Put

$$z = -\pi i,\ (-2\pi i)^2 B = 1 \text{ and } B = -\frac{1}{4\pi^2}$$

Comparing the coefficients of z^3

$$A + C = 0$$

Comparing the constant terms

$$-\pi^3 iA - \pi^2 B + \pi^3 iC - D\pi^2 = 1$$

$$\pi^3 i(C-A) + \frac{1}{2} = 1,$$

$$C - A = \frac{1}{2\pi^3 i}$$

$$2C = \frac{1}{2\pi^3 i},\ C = \frac{1}{4\pi^3 i},\ A = -\frac{1}{4\pi^3 i}$$

$$A = \frac{i}{4\pi^3} \text{ and } C = -\frac{i}{4\pi^3}$$

$$\int_C \frac{e^z dz}{(z^2+\pi^2)^2} = \frac{i}{4\pi^3}\int_C \frac{e^z dz}{(z+\pi i)} - \frac{1}{4\pi^2}\int_C \frac{e^z dz}{(z+\pi i)^2}$$

$$+ \frac{i}{4\pi^3}\int_C \frac{e^z dz}{(z-\pi i)} - \frac{1}{4\pi^2}\int_C \frac{e^z dz}{(z-\pi i)^2}$$

$$[f(z) = e^z,\ f(\pi i) = e^{\pi i} = -1,\ f(-\pi i) = e^{-\pi i} = -1]$$

According to Cauchy's integral formula

$$\int_C \frac{f(z)dz}{(z-a)} = 2\pi i\, f(a),\quad \int_C \frac{f(z)dz}{(z-a)^2} = 2\pi i\, f'(a)$$

$$\int_C \frac{e^z dz}{(z-\pi i)} = -2\pi i,\quad \int_C \frac{e^z dz}{(z+\pi i)} = -2\pi i$$

$$\int_C \frac{e^z dz}{(z-\pi i)^2} = -2\pi i \quad \int_C \frac{e^z dz}{(z+\pi i)} = -2\pi i$$

$$\therefore \quad \int_C \frac{e^z dz}{(z^2+\pi^2)^2} = -2\pi i\left(\frac{i}{4\pi^3} - \frac{1}{4\pi^2} + \frac{i}{4\pi^3} - \frac{1}{4\pi^2}\right)$$

$$= \frac{i}{\pi}.$$

22. Evaluate $\int_C \frac{\cos \pi z^2 dz}{(z-1)(z-2)^3}$ where C is $|z| = 3$ by Cauchy's integral formula.

(JNTU 2003)

Solution: $$\frac{1}{(z-1)(z-2)^3} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2} + \frac{D}{(z-2)^3}$$

$$A = \lim_{z \to 1} \frac{1}{(z-2)^3} = -1$$

$$B = \lim_{z \to 2} \frac{1}{2!} \frac{d^2}{dz^2}\left[\frac{1}{(z-1)}\right] = \lim_{z \to 2} \frac{1}{2} \frac{2}{(z-1)^3} = 1$$

$$C = \lim_{z \to 2} \frac{d}{dz}\left[\frac{1}{(z-1)}\right] = -1$$

$$D = \lim_{z \to 2} \frac{1}{(z-1)} = 1$$

$$\int_C \frac{\cos \pi z^2 dz}{(z-1)(z-2)^3} = \int_C \frac{-\cos \pi z^2 dz}{(z-1)} + \int_C \frac{\cos \pi z^2 dz}{(z-2)}$$

$$-\int_C \frac{\cos \pi z^2 dz}{(z-2)^2} + \int_C \frac{\cos \pi z^2 dz}{(z-2)^3}$$

According to Cauchy's integral formula

$$f^n(a)2\pi i = n! \int_C \frac{f(z)dz}{(z-a)^{n+1}}$$

$$a = 1, f(z) = \cos \pi z^2, f(1) = -1$$

$$\int_C \frac{\cos \pi z^2 dz}{(z-1)} = -2\pi i,\ a = 2$$

$$\int_C \frac{\cos \pi z^2 dz}{(z-2)} = 2\pi i,\ f'(z) = -2\pi z \sin \pi z^2,\ f'(2) = 0$$

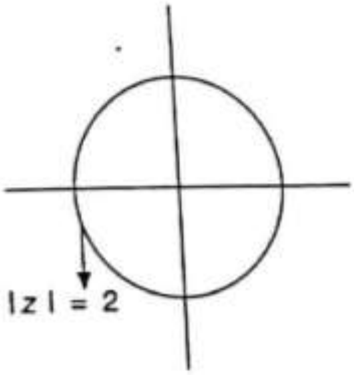
$$f''(z) = -4\pi^2 z^2 \cos \pi z^2 - 2\pi \sin \pi z^2,\ f''(2) = -16\pi^2$$

$$\int_C \frac{\cos \pi z^2 dz}{(z-2)^2} = 0,\ \int_C \frac{\cos \pi z^2 dz}{(z-2)^3} = -16\pi^3 i$$

$$\int_C \frac{\cos \pi z^2 dz}{(z-1)(z-2)^3} = 2\pi i + 2\pi i + 0 - 16\pi^3 i$$

$$= 4\pi i(1 - 4\pi^2).$$

23. Evaluate $\int_C \frac{3z^2 \cosh z}{(z+2i)^2}$ C is $|z-1| = 8$.



Solution: $z = -2i$ is inside $|z-1| = 8$.

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2},\ a = -2i$$

$$f(z) = 3z^2 \cosh z,$$

$$f'(z) = 3z^2 \sinh z + 6z \cosh z$$

$$f'(-2i) = -12 \sinh(-2i) - 12i \cosh(-2i)$$

$$= -12ih (\sin 2 + \cos 2)$$

$$\therefore \int \frac{3z^2 \cosh z\, dz}{(z+2i)^2} = 24\pi (\sin 2 + \cos 2).$$

27. Evaluate $\int_C \frac{(z-3)dz}{(z^2+2z+5)}$ where C is

(a) $|z| = 1$

(b) $|z+1-i| = 2$

(c) $|z+1+i| = 2$.

(JNTU 2004 May)

Solution: $$\frac{1}{z^2+2z+5} = \frac{1}{4i(z+1+2i)} - \frac{1}{4i(z+1-2i)}$$

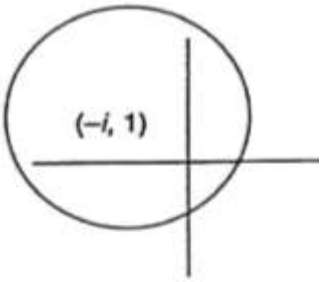
(a) $z = -1 - 2i$ and $-1 + 2i$ both lie outside C

$$\therefore \quad \int_C \frac{(z-3)dz}{(z^2+2z+5)} = 0$$

(b) $$C = |z+1-i| = 2$$

$-1 + 2i$ is inside C whereas $-1 - 2i$ is outside C.

$$\therefore \quad \int \frac{(z-3)dz}{(z+1+2i)} = 0$$



$$\int \frac{(z-3)dz}{(z+1-2i)(z+1+2i)} = \frac{-1}{4i}\int \frac{(z-3)dz}{(z+1-2i)}$$

$$f(z) = z - 3$$

$$a = -1 + 2i$$

$$f(a) = -1 + 2i - 3 = -4 + 2i$$

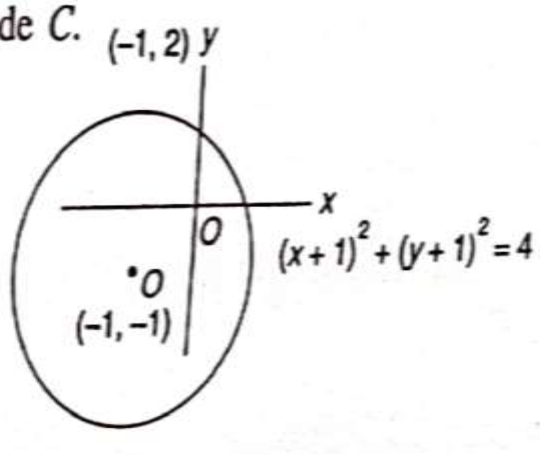
$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)dz}{(z-a)}$$

$$\therefore \quad \int_C \frac{(z-3)dz}{(z^2+2z+5)} = \frac{-1}{4i}\int \frac{(z-3)dz}{(z+1-2i)} = \frac{-1}{4i}(2\pi i)(-4+2i)$$

$$= \pi(2 - i)$$

(c) $|z + 1 + i| = 2$

$-1 + 2i$ is outside whereas $1 - 2i$ is inside C.



$$\therefore \quad \int_C \frac{(z-3)dz}{(z^2+2z+5)} = \frac{1}{4i}\int \frac{(z-3)dz}{(z+1+2i)}$$

$$\int \frac{(z-3)dz}{(z+1-2i)} = 0,$$

$$a = -1 - 2i, f(z) = z - 3$$

$$f(a) = -1 - 2i - 3 = -4 - 2i$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)}$$

$$\int_C \frac{(z-3)dz}{(z^2+2z+5)} = \frac{1}{4i} 2\pi i(-4 - 2i)$$

$$= -\pi(2 + i).$$

32. Evaluate $\int_C \frac{(e^z + z)dz}{(z-1)^4}$ where C is the curve enclosing $z = 1$.

Solution: $z = 1$ is inside C.

According to Cauchy's integral theorem

$$f'''(a) = \frac{3}{\pi i}\int_C \frac{f(z)dz}{(z-a)^4}, a = 1$$

$$f(z) = e^z + z, f'(z) = e^z + 1, f''(z) = e^z, f'''(z) = e^z \text{ and } f'''(1) = e$$

$$\therefore \quad \int_C \frac{(e^z + z)dz}{(z-1)^4} = \frac{\pi i e}{3}.$$

33. $\int_C \frac{(z^2+1)dz}{(z^2-1)}$ where C is $|z| = \frac{3}{2}$.

Solution:

$$\int_C \frac{(z^2+1)dz}{(z^2-1)} = \int_C \left(\frac{z}{(z+1)} + \frac{1}{(z-1)}\right) dz$$

$$= \int_C \frac{z\,dz}{(z+1)} + \int_C \frac{dz}{(z-1)}$$

$$\int_C \frac{z\,dz}{(z+1)},\ z = -1 \text{ inside } C$$

$$f(z) = z$$

According to Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)}$$

$$f(-1) = -1,\ \int_C \frac{z\,dz}{(z+1)} = -2\pi i$$

$$z = 1, f(z) = 1, f(1) = 1$$

$$\int_C \frac{dz}{(z-1)} = 2\pi i$$

$$\therefore \quad \int_C \frac{(z^2+1)dz}{(z^2-1)} = -2\pi i + 2\pi i = 0.$$

34. Evaluate $\int_C \frac{(z^3+z^2+2z-1)dz}{(z-1)^3}$ where C is $|z| = 3$ using Cauchy's integral formula. **(JNTU 2004 Nov.)**

Solution: $f(z) = z^3 + z^2 + 2z - 1$

According to Cauchy's integral formula

$$\pi i\, f''(a) = \int_C \frac{f(z)dz}{(z-a)^3},\ a = 1$$

$$z = 1 \text{ lies inside } C$$

$$f'(z) = 3z^2 + 2z + 2, f''(z) = 6z + 2$$

$$f''(1) = 8$$

$$\therefore \quad \int_C \frac{(z^3+z^2+2z-1)dz}{(z-1)^3} = 8\pi i.$$

36. Evaluate $\int_C \frac{ze^z dz}{(4z+\pi i)^2}$ where C is $|z| = 1$.

Solution: $z = -\frac{\pi i}{4}$ lies inside C.

According to Cauchy's integral formula

$$2\pi i\, f'(a) = \int_C \frac{f(z)dz}{(z-a)^2},\ a = -\frac{\pi i}{4}$$

$$f(z) = ze^z \text{ and } f'(z) = ze^z + e^z$$

$$f'\left(-\frac{\pi i}{4}\right) = -\frac{\pi i}{4}e^{-\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}} = \frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right) - \frac{i}{\sqrt{2}}\left(1+\frac{\pi}{4}\right)$$

$$\int_C \frac{ze^z dz}{4\left(z+\frac{\pi i}{4}\right)^2} = \frac{1}{4}\cdot\frac{2\pi i}{\sqrt{2}}\left[\left(1-\frac{\pi}{4}\right) - i\left(1+\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{2\sqrt{2}}\left[\left(1+\frac{\pi}{4}\right) + i\left(1-\frac{\pi}{4}\right)\right].$$