1) Show that for the fn. $f(z)=\sqrt{xy}$, the C.R eqns are satisfied at the origin but $f'(z)$ does not exist.

$f(z)=\sqrt{xy}=u+iv \quad u=\sqrt{xy}\,,\ v=0$

$u_x=\frac{1}{2}\sqrt{\frac{y}{x}} \quad u_x(0,0)=0 \quad / \ u_y(0,0)=0$

$v_x=0,\ v_y=0 \ ,\ u_x=v_y\,,\quad u_y=-v_x$

Let we calculate $f(z)$ along $mx=y$

$f'(z)=\lim_{x\to 0}\frac{\sqrt{xmx}-0}{x(1+im)}=\frac{\sqrt{m}}{1+im}$ depends on m $\therefore f'(z)$ does not exist

$f'(z)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}$

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2) if $u=x^2-y^2$, $v=\frac{-y}{x^2+y^2}$, then show that both u and v are harmonics but $u+iv$ is not analytic.

$u=x^2-y^2,\ v=\frac{-y}{x^2+y^2} \ / \ u_x=2x \quad \frac{\partial^2 u}{\partial x^2}=2$

$\frac{\partial^2 u}{\partial y^2}=-2 \quad u_y=-2y$

$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

u is harmonic

$\frac{\partial v}{\partial x}=\frac{2xy}{(x^2+y^2)^2}\,,\quad \frac{\partial^2 v}{\partial x^2}=\frac{2y(y^2-x^2)}{(x^2+y^2)^3}$

$\frac{\partial v}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}\,,\quad \frac{\partial^2 v}{\partial y^2}=\frac{2y(x^2-y^2)}{(x^2+y^2)^3}$

$\therefore \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \quad \therefore v$ is harmonic

$f(z) = u + iv = x^2 - y^2 - \frac{iy}{x^2+y^2}$

$u_x \neq v_y \quad u_y \neq -v_x \quad \therefore f(z)$ is not analytic

3) If $w = \log z$, Find $\frac{dw}{dz}$. Determine the point where $f(z)$ is not analytic.

$w = \log z = \log \sqrt{x^2+y^2} + i \tan^{-1}\frac{y}{x}$

$u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}\frac{y}{x}$

$u_x = \frac{x}{x^2+y^2}$, $u_y = \frac{y}{x^2+y^2}$

$v_x = \frac{-y}{x^2+y^2}$, $v_y = \frac{x}{x^2+y^2} \quad \therefore v_x = -u_y$

$u_x = v_y$

C.R eqns are Satisfied $\therefore \frac{dw}{dz} = \frac{1}{z}$ is analytic everywhere

4) Examine if $\bar{z}$ and e^z are analytic fns or not.

a) $\bar{z} \Rightarrow f(z) = x - iy$, $u = x$, $v = -y$, $u_x = 1$, $u_y = 0$

$v_x = 0$, $v_y = -1$ / $u_x \neq v_y$ $\bar{z}$ is not analytic

b) $e^z \Rightarrow f(z) = e^x(\cos y + i\sin y) = e^{x+iy} = e^x e^{iy}$

$u = e^x \cos y$, $v = e^x \sin y$, $u_x = e^x \cos y$

$u_y = -e^x \sin y$, $v_x = e^x \sin y$, $v_y = e^x \cos y$

$v_x = -u_y$, $u_x = v_y$ $\therefore f(z)$ is analytic $f'(z) = e^z$

5) Show that the real and imaginary parts of an analytic fn. $f(z) = u(r,\theta) + i v(r,\theta)$, satisfy Laplace eqn. in polar form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

and $$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

The C-R. eqns in polar form are

$$\underset{(i)}{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}, \quad \underset{(ii)}{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

Differentiate (i) w.r.t r

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial v}{\partial \theta} \longrightarrow ①$$

Differentiate (ii) w.r.t θ

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta} \longrightarrow ②$$

from ① and ②, (i), (ii)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} = 0$$

Similarly we can prove the second relation of $v(r,\theta)$.

Milne - Thomson's method

$$f(z) = u + iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x - iu_y$$

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Considering $z, \bar{z}$ are independent. If $z = \bar{z}$

$$f(z) = u(z,0), iv(z,0)$$

$$f'(z) = \frac{\partial u}{\partial x}(z,0) + i\frac{\partial v}{\partial x}(z,0) = \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)$$

$$f'(z) = \frac{\partial u}{\partial x}(z,0) + i\frac{\partial v}{\partial x}(z,0) = \left(\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right)(z,0)$$

$$f(z) = \int\left[\frac{\partial v}{\partial y}(z,0) + i\frac{\partial v}{\partial x}(z,0)\right]dz$$

i) Integration method

suppose $u(x,y)$ is given.

1) Find u_x, u_y

2) $u_x = v_y$

$$\therefore v = \int v_y\,dy + f(x) = \int u_x\,dy + f(x) \quad \text{①}$$

3) $u_y = -v_x$, $v = -\int u_y\,dx + g(y) \rightarrow$ ②

4) Differentiate ① w.r.t x

$$v_x = \frac{\partial}{\partial x}\left(\int u_x\,dy\right) + f'(x)$$

Comparing with $u_y = -v_x$, we get $f(x)$

5) Differentiate ② w.r.t. y

$$v_y = -\frac{\partial}{\partial y}\left[\int u_y\,dy\right] + g'(y)$$

Comparing with $v_y = u_x$, we get $g(y)$. Substituting with $f(x)$ and $g(y)$, we can get $u(x,y)$ and $v(x,y)$

$$\therefore f(z) = u(x,y) + i\,v(x,y)$$

Milne Thomson's method :

1) $u(x,y)$ is given . find u_x and $u_x(z,0)$
   " u_y " $u_y(z,0)$

2) find $f'(z) = u_x(z,0) - i\,u_y(z,0)$

3) Integrate $f'(z)$ w.r.t z. we get $f(z)$

4) if $v(x,y)$ is given find v_x and $v_x(z,0)$
   " v_y and $v_y(z,0)$

5) find $f'(z) = v_y(z,0) + i\,v_x(z,0)$

6) Integrate $f'(z)$ w.r.t z we get $f(z)$

Exact differential method

If $u(x,y)$ is given

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$$

1) Differentiate (u) w.r.t $y \Rightarrow u_y = -v_x$

2) " (u) " $x \Rightarrow u_x = v_y$

$$dv = -u_y\,dx + u_x\,dy$$

Solve this D.E. which is exact to get $v(x,y)$

$$M = -u_y\,;\; N = v_x$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x^2} = 0$$

Since u is harmonic

$-u_y\,dx + u_x\,dy$ is exact

Among all the 3 methods, Milne Thomson's method is the easiest one.

7) Find an analytic fn. whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}, \qquad f(z) = u + iv \qquad z = x+iy$$

$$u_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - 2\sin 2x \sin 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{z,0} = \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = \frac{2}{\cos 2z - 1}$$

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \quad \therefore \left(\frac{\partial u}{\partial y}\right)_{z,0} = 0$$

According to Milne Thomson's method

$$f'(z) = u_x - i u_y = \frac{-2}{1-\cos 2z}$$

$$f(z) = -2\int \frac{dz}{1-\cos 2z} = -2\int \frac{dz}{2\sin^2 z} = -\int \operatorname{cosec}^2 z \, dz$$

$$f(z) = \cot z + C$$

8) Construct an analytic fn. whose imaginary part is $v(x,y) = 2x(y+1) - 4$ under the restriction $f(1+i) = 2$

$$v(x,y) = 2x(y+1) - 4 \Rightarrow v_x = 2y+2, \ v_y = 2x$$

$f(z)$ is an analytic fn.

$$f'(z) = \frac{\partial v}{\partial y}(z,0) + i\frac{\partial v}{\partial x}(z,0)$$

$$f'(z) = 2z + 2i \Rightarrow f(z) = \int 2(z+i)\,dz$$
$$= z^2 + 2iz + C$$

$$f(1+i) = 2 \Rightarrow (1+i)^2 + 2(1+i)i + C = 4i - 2 + C = 2$$

$c = 4-4i$

$f(z) = z^2 + 2iz + 4(1-i)$

$f(z) = z^2 + 4 + i(2z-4)$

9) Find the analytic fn. whose real part is $e^{-x}(x\cos y + y\sin y)$

$u = e^{-x}(x\cos y + y\sin y) \rightarrow (1)$

Differentiate partially (1) w.r.t (x)

$u_x = e^{-x}\cos y - xe^{-x}\cos y - e^{-x}y\sin y \rightarrow (2)$

Differentiate partially (1) w.r.t (y)

$u_y = -e^{-x}x\sin y + e^{-x}\sin y + ye^{-x}\cos y \rightarrow (3)$

The C-R eqns. are $u_x = v_y$, $u_y = -v_x$

(2) $\Rightarrow$

$\left(\frac{\partial u}{\partial x}\right)_{z,0} = e^{-z} - ze^{-z}$

Milne Thomson's method

$\left(\frac{\partial u}{\partial y}\right)_{z,0} = 0$

$f'(z) = u_x(z,0) - i\,u_y(z,0) = -ze^{-z} + e^{-z}$

$f(z) = \int (e^{-z} - ze^{-z})\,dz = -e^{-z} + ze^{-z} + e^{-z} + c$

$f(z) = ze^{-z} + c$

10) If $u+v = \dfrac{\sinh 2x + \sin 2y}{\cosh 2x + \cos 2y}$. find the analytical fn.

$f(z) = u + iv$

$f(z) = u + iv \qquad i\,f(z) = iu - v$

$$(1+i)\,f(z) = (u-v) + i(u+v)$$
$$= F(z) = U + iV \text{ say}$$

$U = u - v$ is the real part of $F(z)$

$V = u + v$ " " imaginary " " "

$$V = \frac{\sinh 2x + \sin 2y}{\cosh 2x + \cos 2y}$$

$$V_x = \frac{(\cosh 2x + \cos 2y)(2\cosh 2x) - (\sinh 2x + \sin 2y)(2\sinh 2x)}{(\cosh 2x + \cos 2y)^2}$$

$$\frac{\partial V}{\partial x}(z,0) = \frac{(\cosh 2z + 1)(2\cosh 2z) - 2\sinh^2 2z}{(\cosh 2z + 1)^2} = \frac{2}{1 + \cosh 2z}$$

$$V_y = \frac{(\cosh 2x + \cos 2y)(2\cos 2y) - (\sinh 2x + \sin 2y)(-2\sin 2y)}{(\cosh 2x + \cos 2y)^2}$$

$$\frac{\partial V}{\partial y}(z,0) = \frac{(\cosh 2z + 1)(2)}{(\cosh 2z + 1)^2} = \frac{2}{1 + \cosh 2z}$$

$$F'(z) = \frac{\partial V}{\partial y}(z,0) + i\,\frac{\partial V}{\partial x}(z,0) = \frac{2(1+i)}{1 + \cosh 2z}$$

$$F(z) = 2(1+i)\int \frac{dz}{1 + \cosh 2z} = (1+i)\int \frac{dz}{\cosh^2 z}$$

$$= (1+i)\int \operatorname{sech}^2 z\, dz = (1+i)\tanh z + C$$

$$f(z) = \frac{F(z)}{1+i} = \tanh z + C$$

Milne-Thomson method for finding a holomorphic function

In mathematics, the **Milne-Thomson method** is a method of finding a holomorphic function, whose real or imaginary part is given.[1] The method greatly simplifies the process of finding the holomorphic function whose real or imaginary part is given. It is named after Louis Melville Milne-Thomson.

Method for finding the holomorphic function

Let $f(z) = u(x,y) + iv(x,y)$ be any holomorphic function.

Let $z = x + iy$ and $\bar{z} = x - iy$ where x and y are real.

Hence,

$$x = \frac{z + \bar{z}}{2}$$

$$\text{and } y = \frac{z - \bar{z}}{2i}$$

Therefore, $f(z) = u(x,y) + iv(x,y)$ is equal to

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

This can be regarded as an identity in two independent variables z and $\bar{z}$. We can therefore, put $z = \bar{z}$ and get $f(z) = u(z,0) + iv(z,0)$

So, $f(z)$ can be obtained in terms of z simply by putting $x = z$ and $y = 0$ in $f(z) = u(x,y) + iv(x,y)$ when $f(z)$ is a holomorphic function.

Now, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

Since, $f(z)$ is holomorphic, hence Cauchy–Riemann equations are satisfied. Hence, $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$.

Let $\frac{\partial u}{\partial x} = \Phi(x,y)$ and $\frac{\partial u}{\partial y} = \Psi(x,y)$.

Then

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = \Phi(x,y) - i\Psi(x,y)$$

Now, putting $x = z$ and $y = 0$ in the above equation, we get

$$f'(z) = \Phi(z,0) - i\Psi(z,0).$$

Integrating both sides of the above equation we get

$$\int f'(z)\,dz = \int \Phi(z,0)\,dz - i\int \Psi(z,0)\,dz$$

or

$$f(z)=\int f'(z)\,dz=\int \Phi(z,0)\,dz-i\int \Psi(z,0)\,dz+c$$

which is the required holomorphic function.

Example

Let $u(x,y)=x^4-6x^2y^2+y^4$, and let the desired holomorphic function be $f(z)=u(x,y)+iv(x,y)$

Then as per the above process we know that

$$f'(z)=\frac{\partial u(x,y)}{\partial x}+i\frac{\partial v(x,y)}{\partial x}.$$

But as $f(z)$ is holomorphic, so it satisfies Cauchy–Riemann equations.

Hence, $\dfrac{\partial u(x,y)}{\partial x}=\dfrac{\partial v(x,y)}{\partial y}$ and $\dfrac{\partial u(x,y)}{\partial y}=-\dfrac{\partial v(x,y)}{\partial x}$

Or $u_x=v_y$ and $u_y=-v_x$.

Substituting these values in $f'(z)$ we get,

$$f'(z)=\frac{\partial u(x,y)}{\partial x}-i\frac{\partial u(x,y)}{\partial y}$$

Hence,

$$f'(z)=(4x^3-12xy^2)-i(-12x^2y+4y^3)$$

This can be written as $f'(z)=\Phi(x,y)-i\Psi(x,y)$

where, $\Phi(x,y)=(4x^3-12xy^2)$ and $\Psi(x,y)=-12x^2y+4y^3$.

Rewriting $f'(z)=\Phi(x,y)-i\Psi(x,y)$ using $x=z$ and $y=0$

$$f'(z)=4z^3-i(0)$$

Integrating both sides w.r.t dz we get,

$$\int f'(z)\,dz=\int 4z^3\,dz+\int 0\,dz$$

Hence, $f(z)=z^4+c$ is the required holomorphic function.

References

1. Milne-Thomson, L. M. (July 1937). "1243. On the Relation of an Analytic Function of z to Its Real and Imaginary Parts". *The Mathematical Gazette*. **21** (244): 228. doi:10.2307/3605404 (https://doi.org/10.2307%2F3605404). JSTOR 3605404 (https://www.jstor.org/stable/3605404).