Damietta University
Faculty of Science
Department of Mathematics

Third Year Stats. & Comp.
Stochastic Process
Date: Sunday 22-3-2020
Time: 2 hours, from 10 to 12.

Lecture # 6

Discrete-Time Markov Chains

Topic: Classification of States

You should know

1- how to classify the states of discrete MC,
2- the properties of the two relations: $i \to j$ & $i \leftrightarrow j$,
3- and understood the concept of each of: Accessible state - communicate and communicating class- closed set- irreducible MC- absorbing state- reflecting state- period, periodic and aperiodic state- transient state- recurrent state- mean recurrence time- null recurrent state- positive recurrent state- ergodic MC.

Prof. Dr. M A El-Shehawey

<u>Lecture # 6</u>
<u>Classification of States</u>

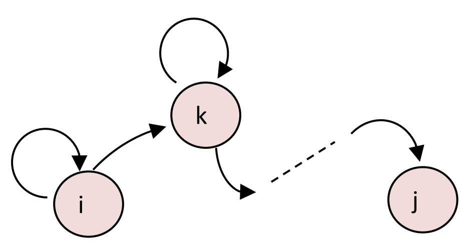
The main interest in **Markov Chain** (MC) is to obtain the limiting probability that the state will be in *j* given that the initial state is in *i*, that is what is $\lim_{n\to\infty} \Pr\left(X_n = j \mid X_0 = i\right)$?
The limiting probability does not always exit, but it exists when the transition probability matrix (TPM) of a MC satisfies certain properties.

To understand the *n*-step transition in more detail, we need to study how mathematicians classify the states of a MC. States of MC's are classified by the digraph representation (Some time omitting the actual probability values).

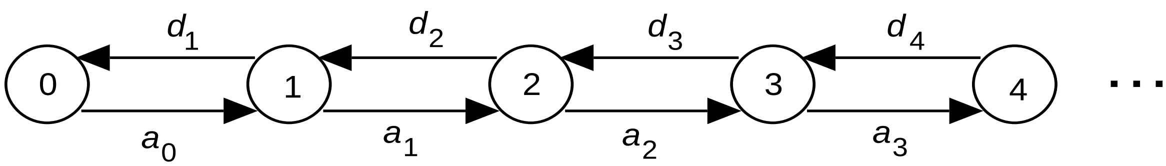
<u>Basic Concepts</u>
The following concepts are very importance.

- **<u>Def (1) (</u>**<u>Path</u>**<u>)</u>:** Given two states *i* and *j*, a **path** from *i* to *j* is a sequence of states, where each transition has a positive probability of occurring.

- **<u>Def (2) (</u>**<u>Reachable</u>**<u>):</u>** A state *j* is called **reachable** (or **accessible**) from state *i* iff $p_{ij}^{(n)} > 0$ for some $n \geq 0$, i.e., if there is a path leading from *i* to *j*. Write $i \to j$.



- **<u>Def (3) (</u>**<u>Communicate</u>**<u>):</u>** Two states *i* and *j* are said to **communicate** iff *j* is reachable from *i*, and *i* is reachable from *j* (i.e., the states *i* and *j* are accessible from each other; that is there exists two integers n and m such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$). Write $i \leftrightarrow j$.



<u>Properties of the Relation of Communication</u>

<u>Lemma (1)</u>.

The relation of communication satisfies the following three properties: For all *i*, *j*, and *k*:

a) $(i \leftrightarrow i)$: State *i* communicate with itself.

b) $(i \leftrightarrow j) \Rightarrow (j \leftrightarrow i)$: State *i* communicates with *j*, then state *j* communicates with *i*.

c) $(i \leftrightarrow j)$ and $(j \leftrightarrow k) \Rightarrow (i \leftrightarrow k)$: State *i* communicates with *j*, and state *j* communicates with state k, then *i* communicates with k.

Proof. The proof of properties a) and b) are obvious (leave to students).
c) Suppose that $i \leftrightarrow j$ and $j \leftrightarrow k$. Then, since $i \rightarrow j$ and $j \rightarrow k$ then there exit some $m \geq 0$ and $n \geq 0$ for which $p_{ij}^{(m)} > 0$, $p_{jk}^{(n)} > 0$. So,

$$p_{ik}^{(m+n)} = \sum_{l} p_{il}^{(m)} p_{lk}^{(n)} \geq p_{ij}^{(m)} p_{jk}^{(n)} > 0.$$

That is, there exits $r \geq 0$ such that $p_{ik}^{(r)} > 0$. Hence, $i \rightarrow k$. Similarly, we can show that there exits $s \geq 0$ such that $p_{ki}^{(s)} > 0$, which states that $k \rightarrow i$. Therefore, $i \leftrightarrow k$.

- A set of states C is a **communicating class** **(class)** if every pair of states in C communicates with each other, and no state in C communicates with any state not in C.
- Two states that communicate are said to be in the same class.
- Two classes are either identical or disjoint (have no communicating states).

For example, the classes of the MC $\{X_n, n \geq 0\}$ on the state space $SS = \{0,1,2,3\}$ with the following TPM are $\{0,1\}, \{2\}$, and $\{3\}$:

$$\mathbf{M} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

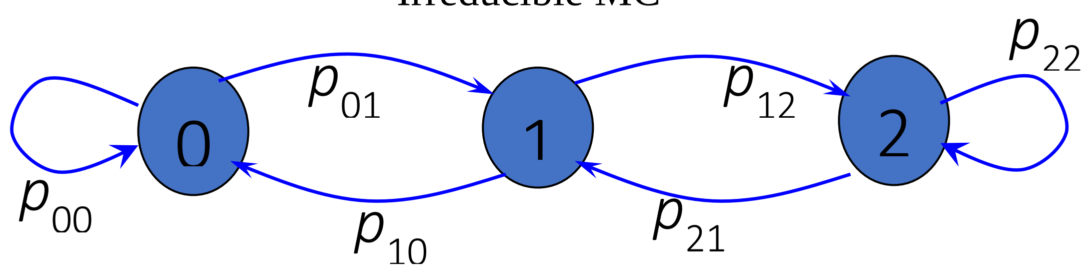
- **Def (4) (Closed set):** A set of states, C in a MC is called a **closed set** if no state outside of C is reachable from any state in C. (i.e. if no one-step transition is possible from a state in C to a state outside of C. That is $\forall\, i \in C$ and all $j \notin C$, $p_{ij} = 0$. In fact, $p_{ij}^{(n)} = 0$ for all $n \geq 1$).

- **Def (5) (Irreducible set):** A closed set C of states is irreducible if any state $j \in C$ is **reachable** from every state $i \in C$.

- **Def (6) (Irreducible MC)**: A MC is said to be **irreducible** if there is only one class (all states communicate with each other). This implies that n exists such that $\Pr\left(X_n = j \middle| X_0 = i\right) = p_{ij}^{(n)} > 0, \forall i, j$.

Thus, A MC with state space SS is said to be **irreducible** if $i \leftrightarrow j$ for all $i, j \in SS$.

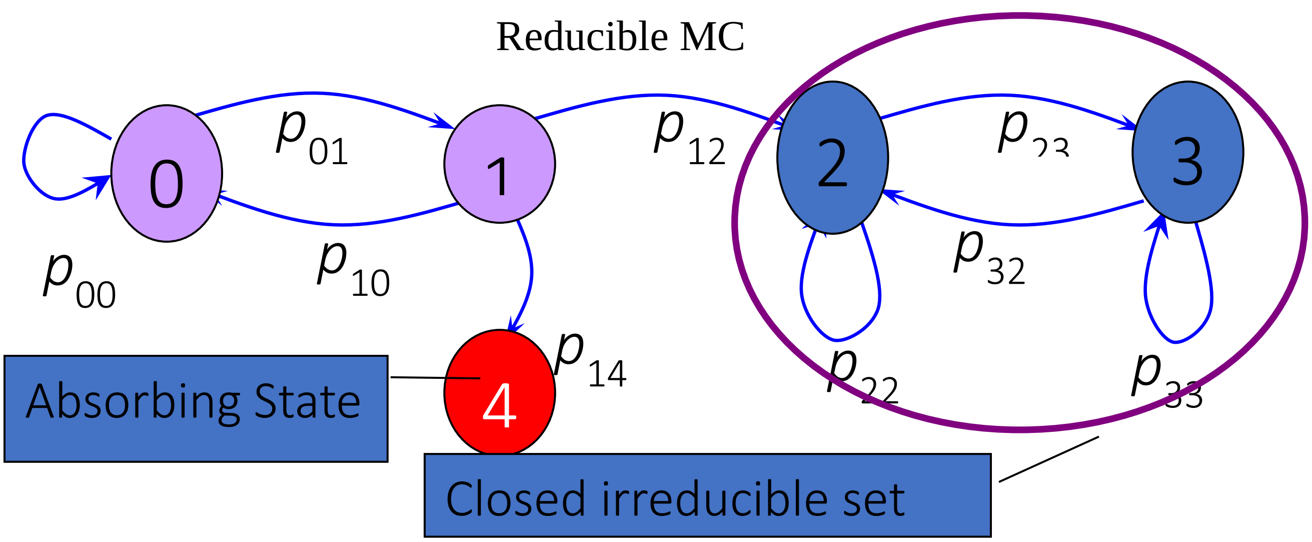
Irreducible MC



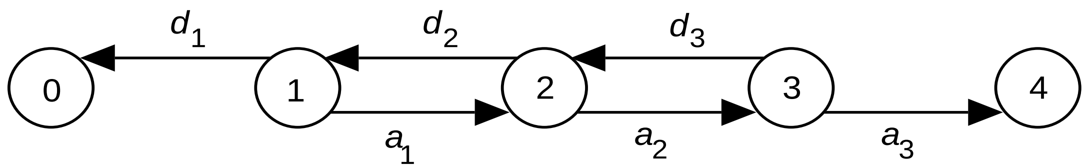
For example, the MC $\{X_n, n \geq 0\}$ on the state space $SS = \{0,1,2\}$ with the following TPM is **irreducible**

$$\mathbf{M} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 1/4 \\ 0 & 1/3 & 2/3 \end{pmatrix}.$$

Reducible MC



- **<u>Def (7) (</u>**<u>Absorbing</u>**<u>):</u>** A state i that is never left after it is entered is said to an **absorbing state,** i.e., if $p_{ii} = 1$.



- In this diagram there are two absorbing states: 0 and 4.
- Note that, an absorbing state itself is classified as one class.

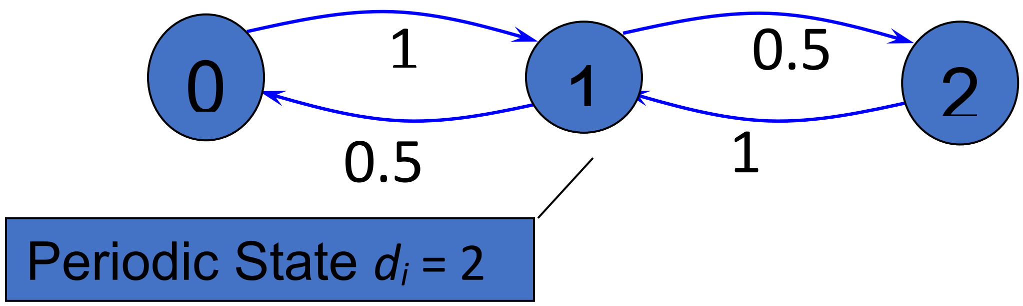
- **Def (8)** (Reflecting state): Reflecting state is the state at which the MC will continue, but in the reverse direction.
- **Def (9)** (Partially Reflecting state): Partially Reflecting state is the state which is the same as the reflecting state, except that the MC will stay on that state for a unit step time with some probability.
- **Def (10) (**Period state**):** State i is said to have period d_i if $p_{ii}^{(n)} = 0$ whenever n is not divisible by d_i, and d_i is the greatest common divisor of all integers $n \geq 1$:
- $p_{ii}^{(n)} > 0$. If the only possible steps in which state i can occur again are $d_i, 2d_i, 3d_i, \ldots$. In that case, the recurrence time for state i has period d_i.
- **Def (11) (**Periodic and aperiodic**):** A state i with period $d_i > 1$ is said to be **periodic**, that is a state is **periodic** if it can only return to itself after a fixed number of transitions greater than 1 (or multiple of a fixed number).

If a state is not periodic, it is referred to as **aperiodic**: $d_i = 1$.

A MC in which every state is a **periodic** is known as a periodic MC.

Periodicity is a class property (= state *i* has period d_i and $(i \leftrightarrow j)$ then *j* has period d_i).

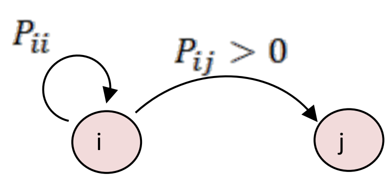
<u>For example</u>, for the MC $\{X_n, n \geq 0\}$ on the state space $SS = \{0,1,2\}$,



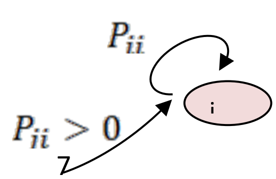
The TPM is:

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

- **<u>Def (12) (</u>**<u>Transient state</u>**<u>):</u>** A state *i* is said to be a **transient state** if there exists a state *j* that is reachable from *i*, but the state *i* is not reachable from state *j*. (i.e., if the probability of MC eventually ever returns to state *i* is less than 1).



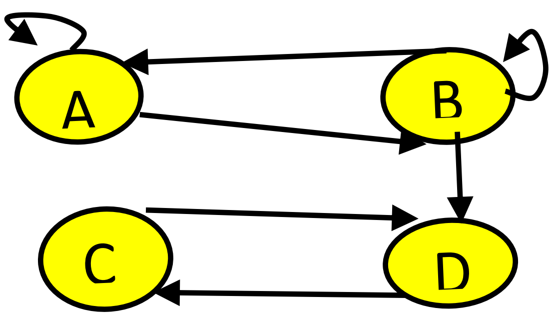
- **<u>Def (13) (</u>**<u>Recurrent state</u>**<u>):</u>** A state *i* is said to be a **recurrent** (**persistent** "not transient") if, the probability of MC eventually ever returns to state *i* equal one (i.e. if it is revisited infinitely often with probability one).



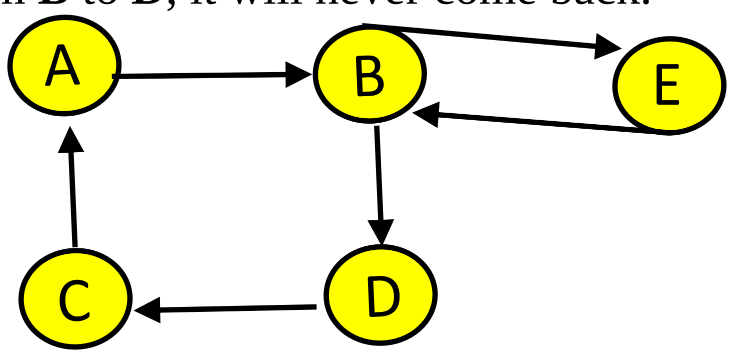
A state is either **recurrent** state or **transient** state. For a recurrent state, it can be either absorbing state or non-absorbing state. The state space of a MC, can be partitioned into a transient set and closed sets of recurrent states (possibly absorbing).

If a MC has **finite state space**, then at least one of the states is **recurrent**.

A class is either **all recurrent** or **all transient** and may be all periodic or **aperiodic.** All states in an **irreducible** MC are **recurrent**.



A and **B** are ***transient*** states**,** **C** and **D** are ***recurrent*** states. Once the MC moves from **B** to **D**, it will never come back.



A MC is **irreducible** if the corresponding graph is connected (and thus all its states are recurrent).
A MC is ***periodic*** if all the states in it have a period $k > 1$. The above MC has **period** 2.

<u>For example</u>, For the MC $\{X_n, n \geq 0\}$ on the state space

(1)- $SS = \{0,1,2,3\}$ with the following TPM:

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

All states **communicate**. Therefore, all states are **<u>recurrent</u>**.

(2)- $SS = \{0,1,2,3,4\}$ with the following TPM:

$$\mathbf{M} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{pmatrix},$$

There are **three classes** {0, 1}, {2, 3} and {4}. The first two are **recurrent** and the third is **transient**.

(3)- $SS = \{0,1,2,3\}$ with the following TPM, with x is a prob.

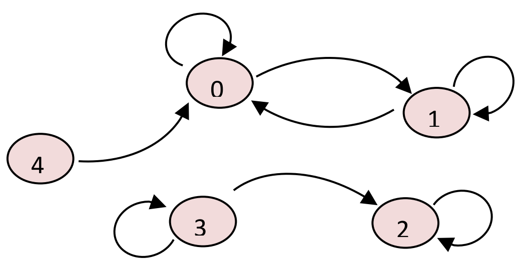
$$\mathbf{M} = \begin{pmatrix} 0 & x & x & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ x & 0 & 0 & x \end{pmatrix}.$$

Every pair of states that **communicates** forms a single **recurrent class**; however, the states are **<u>not periodic</u>**.

Thus, the MC is **aperiodic** and **irreducible**.

(4)- $SS = \{0,1,2,3,4\}$ with the following TPM, with x is a prob.

$$\mathbf{M} = \begin{pmatrix} x & x & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & x & x & 0 \\ x & 0 & 0 & 0 & 0 \end{pmatrix},$$



States 0 and 1 **communicate** and form a **recurrent class**.
States 3 and 4 form separate **transient classes**.
State 2 is an **absorbing** state and forms a **recurrent class**.

(5)- $SS = \{0,1,2,3\}$ with the following TPM, with x is a prob.

$$\mathbf{M} = \begin{pmatrix} 0 & x & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \\ x & 0 & 0 & 0 \end{pmatrix}.$$

Every state communicates with every other state, so we have **irreducible** MC. This MC is **irreducible** and **periodic**.

(6)- $SS = \{1,2,3,4,5\}$ with the following TPM, with x is a prob.

$$\mathbf{M} = \begin{pmatrix} x & x & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ 0 & 0 & x & x & 0 \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{pmatrix},$$

State 2 is **accessible** from state 1, states 3 and 4 are **accessible** from state 5, but states 3 is not accessible from state 2.
States 1 & 2 **communicate**; states 3, 4 & 5 **communicate**; states 1,2 and states 3,4,5 do not **communicate**.
States 1 & 2 form one **communicating class**.
States 3, 4 & 5 form another **communicating class**.
This MC is not an **irreducible.**

- **Def (14)** (Mean recurrence time): The mean recurrence time h_i is defined as the average time for first return to state i. (For a transient state, $h_i \equiv \infty$).
- There are two types of recurrent states:
- **Def (15)** (Null recurrence state): A recurrence state i is called **Null recurrent** state if the expected time to return to the state i is infinite (the mean recurrence time $h_i = \infty$); otherwise it is called **Non-null recurrent** (or **Positive recurrent**) state.
- **Def (16)** (Non-null recurrence state): A recurrence state i is called **Non-null** (or **Positive) recurrent** state if the expected time to return to the state is finite (if the mean recurrence time $h_i < \infty$).
- **Def (17)** (Ergodic state): A recurrent state is said to be **ergodic** state if it is both **non-null recurrent** and **a periodic**. This implies that it is possible to come back to that state in any given number of steps and that such a return will always occur with finite mean recurrence times.

- **Def (18)** (Ergodic MC): If all states in a MC are ergodic it is called ergodic MC (i.e. all states in a MC are non-null recurrent, a periodic).

Note that an a periodic, irreducible "all states communicate with each other", MC with a finite number of states will always be ergodic.

To determine if a state is recurrent or not, the following result would be useful instead of using the definition.

Theorem (1). State j is **recurrent** if and only if $\sum_{n=1}^{\infty} p_{jj}^{(n)} = \infty$.

Proof. To prove the **sufficiency**, suppose that the state j is **recurrent**. Then, the number of visits (# visits) to state j is **infinite** and so the expected number of visits to state j is also infinite. That is

$$E\left[\# \text{ visits to } j \mid X_0 = j\right] = \infty.$$

Let us define the indicator variable: $I_n = \begin{cases} 1 \text{ if } X_n = j \\ 0 \text{ if } X_n \neq j \end{cases}$.

Then $\sum_{n=1}^{\infty} I_n$ reduces to the number of visits to state j.

Hence, the **expected** number of visits to state j given $X_0 = j$ is

$$\begin{aligned} E\left[\# \text{ visits to } j \mid X_0 = j\right] &= E\left[\sum_{n=1}^{\infty} I_n \mid X_0 = j\right] = \sum_{n=1}^{\infty} E\left[I_n \mid X_0 = j\right] \\ &= \sum_{n=1}^{\infty} \Pr\left(X_n = j \mid X_0 = j\right) = \sum_{n=1}^{\infty} p_{jj}^{(n)} \end{aligned}$$

The necessity follows immediately.

Lemma (1). State j is **transient** if $\sum_{n=1}^{\infty} p_{jj}^{(n)} < \infty$. (leave to students).

❖ In a finite-state MC not all states are transient, i.e., there exist at least one recurrent state.

Theorem (2). If state i is recurrent and $i \leftrightarrow j$, then state j is recurrent.

Proof. Suppose that state i is recurrent and $i \leftrightarrow j$. Then there exist $m > 0$ and $n > 0$ such that $p_{ij}^{(m)} > 0$, and $p_{ji}^{(n)} > 0$. Also, from Chapman-Kolmogorov equation the following holds for $l > 0$: $p_{jj}^{(m+n+l)} \geq p_{ji}^{(m)} p_{ii}^{(l)} p_{ij}^{(n)}$. So,

$$\sum_{l=1}^{\infty} p_{jj}^{(m+n+l)} \geq p_{ji}^{(m)} p_{ij}^{(n)} \sum_{l=1}^{\infty} p_{ii}^{(l)} = \infty .$$

Questions. Prove that

- If $i \to j$ and $j \to k$ then $i \to k$.
- If i is **recurrent** and $i \to j$, prove that state j is **recurrent**.
- If state i is **transient**, and state i **communicates** with state j, prove that the state j is **transient** → transience is also a **class property**.