Fourth Year Stats. & Comp.
Combinatorics
Date: Saturday 21-3-2020
Time: One hour

Damietta University
Faculty of Since
Department of Mathematics

Lecture # 6

Topic: Generalized Fibonacci sequences

You should know

a- how to solve the second order recurrence relation, in the two cases:

1- by some changing the recurrence relations slightly while preserving the first two initial terms of the sequence,
2- by some altering the initial terms of the sequence but maintaining the recurrence relations.

b- study on some properties of particular sequences

Prof. Dr. M A El-Shehawey

Lecture # 6
Generalized Fibonacci sequences

Some Aspects of Sequences

Consider the m^{th} order homogeneous linear recurrence relation

$$T_n = a_1 T_{n-1} + a_2 T_{n-2} + ... + a_m T_{n-m} = \sum_{i=1}^{m} a_i T_{n-i},$$

where $a_1, a_2, \cdots, a_m$ are real constants, $a_m \neq 0$. To generate a sequence $\{T_n\}_{n=0}^{\infty}$, we specify initial values $T_0, T_1, ..., T_{m-1}$.

Generalized Fibonacci sequences

The classical Fibonacci sequence can be generalized in several ways, either by some changing the recurrence relations slightly while preserving the first two initial terms of the sequence, or by some altering the initial terms of the sequence but maintaining the recurrence relations.

❖**Generalized** $(a,b;p,q)$**-Fibonacci sequence** $\{T_n = T_n(a,b;p,q)\}$

For $m=2$ and $n \geq 0$, we write the notation $T_n(a,b;p,q)$, or briefly T_n, as the generalized Fibonacci sequence $\{T_n\}_{n=0}^{\infty}$ which is defined by the 2^{ed} order homogeneous linear recurrence relation

$$T_n = pT_{n-1} + qT_{n-2}, \text{ for all } n \geq 2 \qquad \text{(I)}$$

with the initial conditions $T_0 = a$ and $T_1 = b$, where a,b are arbitrary integers.

If $q \neq 0$, as well as $p^2 + 4q \neq 0$. $\{T_n = T_n(a,b;p,q)\}$ is called **Horadam** sequence. It generalizes many sequences. Examples of

such sequences are Fibonacci numbers sequence $\{F_n\}_{n\geq 0}$, Lucas numbers sequence $\{L_n\}_{n\geq 0}$, Pell numbers sequence $\{P_n\}_{n\geq 0}$, Pell-Lucas numbers sequence $\{Q_n\}_{n\geq 0}$, Jacobsthal numbers sequence $\{J_n\}_{n\geq 0}$ and Jacobsthal-Lucas numbers sequence $\{j_n\}_{n\geq 0}$.
The characteristic equation of recurrence relation (I) is

$$1-px-qx^2=0.$$

The two roots of this characteristic equation are

$$\alpha=\frac{p+\sqrt{p^2+4q}}{2} \text{ and } \beta=\frac{p-\sqrt{p^2+4q}}{2}.$$

<u>Theo (1)</u>. The **generating function** for the generalized$(a,b;p,q)$-Fibonacci sequence $\{T_n=T_n(a,b;p,q)\}$ is given by

$$G(x)=\frac{a+(b-ap)x}{1-px-qx^2}.$$

Proof. Let $G(x)=\sum_{n=0}^{\infty}T_nx^n$ be the generating function of the generalized -Fibonacci sequence $\{T_n=T_n(a,b;p,q)\}$, which can be obtained through the usual trick, we note that $T_0=a$ and $T_1=b$.
We write

$$G(x)=T_0+T_1x+T_2x^2+...+T_nx^n+...,$$

then obtain $pxG(x)$and $qx^2G(x)$, subtract them from $G(x)$, use the recurrence to get remove of all summands except the first two and impose the initial conditions.
Using the rational expansion theorem, we get the Binet's form of the **Horadam sequence**:

$$T_n = A\alpha^n + B\beta^n = A\left(\frac{p+\sqrt{p^2+4q}}{2}\right)^n + B\left(\frac{p-\sqrt{p^2+4q}}{2}\right)^n$$
$$= \left(\frac{b-\beta a}{\alpha-\beta}\right)\alpha^n + \left(\frac{a\alpha-b}{\alpha-\beta}\right)\beta^n,$$

where $A = \frac{b-\beta a}{\sqrt{p^2+4q}}$, and $B = \frac{a\alpha-b}{\sqrt{p^2+4q}}$.

- **Particular Cases**

(1)- **Fibonacci** **Sequence** $\{T_n = T_n(0,1;p,q)\}$:

If we set $a=0$, and $b=1$, we obtain the a **Fibonacci sequence** $\{T_n = T_n(0,1;p,q)\}$, in this case

$$T_n = pT_{n-1} + qT_{n-2} \text{, for all } n \geq 2 \qquad (1)$$

with the initial conditions $T_0 = 0$ and $T_1 = 1$.

Theo (1). The generating function for the Fibonacci sequence $\{T_n = T_n(0,1;p,q)\}$ is given by, $G(x) = \frac{x}{1-px-qx^2}$.

Proof. Let $G(x) = \sum_{n=0}^{\infty} T_n x^n$ be the generating function of the sequence $\{T_n = T_n(0,1;p,q)\}$, we note that $T_0 = 0$ and $T_1 = 1$. Thus

$$G(x) = T_0 + T_1x + T_2x^2 + \ldots + T_nx^n + \ldots$$
$$pxG(x) = pT_0x + pT_1x^2 + pT_2x^3 + \ldots + pT_nx^{n+1} + \ldots$$
$$qx^2G(x) = qT_0x^2 + qT_1x^3 + qT_2x^4 + \ldots + qT_nx^{n+2} + \ldots.$$

We will add the power series $G(x)$, $-pxG(x)$, and $-qx^2G(x)$ we have

$$G(x)-pxG(x)-qx^2G(x)=T_0+(-pT_0+T_1)x+(-qT_0-pT_1+T_2)x^2+\dots$$

Notice that if we take our rearranged recursion formula

$$T_n-pT_{n-1}-qT_{n-2}=0,$$

with $n=2$ we get $T_2-pT_1-qT_0=0$.

Thus, the coefficient of x^2 term in our combined series is zero. In fact, using the recursion formula, the coefficient of the terms after the x^2 term we see they are all zero. Thus, we have

$$G(x)-pxG(x)-qx^2G(x)=T_0+(-pT_0+T_1)x.$$

Since $T_0=0$ and $T_1=1$, then

$$G(x)-pxG(x)-qx^2G(x)=x,$$

and

$$G(x)\left(1-px-qx^2\right)=x\Rightarrow G(x)=\frac{x}{1-px-qx^2}=\sum_{n=0}^{\infty}T_nx^n.$$

Theo (2). (Binet's Formula**)**

The terms of the sequence $\{T_n=T_n(0,1;p,q)\}$ are given by

$$T_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}=\frac{1}{\sqrt{p^2+4q}}\left\{\left(\frac{p+\sqrt{p^2+4q}}{2}\right)^n-\left(\frac{p-\sqrt{p^2+4q}}{2}\right)^n\right\},$$

where $\alpha=\dfrac{p+\sqrt{p^2+4q}}{2}$ and $\beta=\dfrac{p-\sqrt{p^2+4q}}{2}$ are the roots of the characteristic equation $1-px-qx^2=0$.

Proof. We express a function $G(x)=\sum_{n=0}^{\infty}T_n x^n$ for T_n as a sum of partial fractions. Let $1-px-qx^2=(1-\alpha x)(1-\beta x)$, and consider

$$G(x)=\frac{x}{1-px-qx^2}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x},$$

then, $x=A(1-\beta x)+B(1-\alpha x)$. If $x=\frac{1}{\beta}$, then

$$\frac{1}{\beta}=B\left(1-\frac{\alpha}{\beta}\right)\to\frac{1}{\beta}=B\left(\frac{\beta-\alpha}{\beta}\right)\to B=\frac{1}{\beta-\alpha}\Rightarrow B=\frac{-1}{\alpha-\beta}$$

Similarly, if $x=\frac{1}{\alpha}$, then we have $\frac{1}{\alpha}=A\left(1-\frac{\beta}{\alpha}\right)\Rightarrow A=\frac{1}{\alpha-\beta}$, then

$$G(x)=\frac{x}{1-px-qx^2}=\frac{1/(\alpha-\beta)}{1-\alpha x}+\frac{-1/(\alpha-\beta)}{1-\beta x}=\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\alpha^n x^n$$

$$-\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\beta^n x^n=\sum_{n=0}^{\infty}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)x^n=\sum_{n=0}^{\infty}T_n x^n\Rightarrow T_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},$$

it's Binet's formula for the Fibonacci sequence $\{T_n=T_n(0,1;p,q)\}$.

Theo (3). Sum of the first n terms of the Fibonacci sequence $\{T_n=T_n(0,1;p,q)\}$ is

$$\sum_{i=1}^{n}T_i=\frac{T_{n+1}+qT_n-1}{p+q-1}.$$

Proof. By summing up the geometric partial sums,

$$\sum_{k=0}^{n}\alpha^k=\frac{1-\alpha^{n+1}}{1-\alpha},$$

analogously for β and $1-p-q=(1-\alpha)(1-\beta)$, we have

$$\sum_{k=0}^{n} T_k = \frac{1}{\alpha-\beta}\sum_{k=0}^{n}\alpha^k - \frac{1}{\alpha-\beta}\sum_{k=0}^{n}\beta^k = \frac{1-qT_n-T_{n-1}}{1-p-q}.$$

Theo (4). Sum of the first n terms with **odd** indices is

$$\sum_{i=1}^{n} T_{2i-1} = \frac{1}{1-q}\left[1+p\left(T_2+T_4+...+T_{2n}\right)-T_{2n+1}\right].$$

This identity becomes

$$T_{2n+1}-1 = p\left(T_0+T_2+T_4+...+T_{2n-2}\right)+\left(q-1\right)\left(T_1+T_3+T_5+...+T_{2n-1}\right).$$

Theo (5). Sum of the first n terms with **even** indices is

$$\sum_{i=1}^{n} T_{2i} = \frac{1}{1-q}\left[p\left(T_1+T_3+...+T_{2n+1}\right)-T_{2n+2}\right].$$

This identity becomes

$$T_{2n+2} = \left(q-1\right)\left(T_0+T_2+T_4+...+T_{2n}\right)+p\left(T_1+T_3+T_5+...+T_{2n+1}\right).$$

Theo (6). Multiplication of two consecutive Fibonacci sequence $\left\{T_n = T_n\left(0,1;p,q\right)\right\}$ is given by $T_nT_{n+1} = p\sum_{k=0}^{n} q^{n-k}T_k^2$.

Proof. Since $T_n = pT_{n-1}+qT_{n-2}$, and $T_{n+1} = pT_n+qT_{n-1}$. Then

$$\begin{aligned}
T_nT_{n+1} &= T_n\left(pT_n+qT_{n-1}\right) = pT_n^2+qT_nT_{n-1} = pT_n^2+q\left(pT_{n-1}+qT_{n-2}\right)T_{n-1} \\
&= pT_n^2+q\left(pT_{n-1}^2+qT_{n-1}T_{n-2}\right) = pT_n^2+pqT_{n-1}^2+q^2T_{n-1}T_{n-2} \\
&= pT_n^2+pqT_{n-1}^2+q^2\left(pT_{n-2}^2+qT_{n-2}T_{n-3}\right) \\
&= pT_n^2+pqT_{n-1}^2+pq^2T_{n-2}^2+q^3T_{n-2}T_{n-3}+... \\
&= pT_n^2+pqT_{n-1}^2+pq^2T_{n-2}^2+...+pq^{n-1}T_1^2+q^n\left[pT_0^2+qT_0T_{-1}\right] \\
&= pT_n^2+pqT_{n-1}^2+pq^2T_{n-2}^2+...+pq^{n-1}T_1^2+pq^nT_0^2 \\
&= p\left(T_n^2+qT_{n-1}^2+q^2T_{n-2}^2+...+q^{n-1}T_1^2+q^nT_0^2\right) = p\sum_{k=0}^{n} q^{n-k}T_k^2.
\end{aligned}$$

<u>Theo (7)</u>. For the matrix $\mathbf{U}=\begin{pmatrix}0 & 1\\ q & p\end{pmatrix}$, we have $\mathbf{U}^n=\begin{pmatrix}qT_{n-1} & T_n\\ qT_n & T_{n+1}\end{pmatrix}$.

Proof: (Using principal mathematical induction)

For $n=1$, then $\mathbf{U}=\begin{pmatrix}qT_0 & T_1\\ qT_1 & T_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ q & p\end{pmatrix}$, the result is true.

Suppose the result is true for $n=k$, i.e., let

$\mathbf{U}^k=\begin{pmatrix}qT_{k-1} & T_k\\ qT_k & T_{k+1}\end{pmatrix}$ be true. Now

$$\mathbf{U}^{k+1}=\mathbf{U}^k\mathbf{U}=\begin{pmatrix}qT_{k-1} & T_k\\ qT_k & T_{k+1}\end{pmatrix}\begin{pmatrix}0 & 1\\ q & p\end{pmatrix}=\begin{pmatrix}qT_k & qT_{k-1}+pT_k\\ T_{k+1} & qT_k+pT_{k+1}\end{pmatrix}=\begin{pmatrix}qT_k & T_{k+1}\\ qT_{k+1} & T_{k+2}\end{pmatrix}.$$

Thus, the result is true for $n=k+1$.

Note that $|\mathbf{U}|=-q$.

<u>Theo (8)</u>. For any positive integers, p,q, m, and n, $0\le m\le n$:

$$T_{m+n}=T_mT_{n+1}+qT_{m-1}T_n.$$

Proof: Using principal mathematical induction.

For $n=1$, we have $T_{m+1}=T_mT_2+qT_{m-1}T_1$, since $T_0=0$, $T_1=1$ and $T_2=p$. Then $T_{m+1}=pT_m+qT_{m-1}$. Thus, the result is true for $n=1$.

Suppose the result is true for $n=k$, i.e., let

$T_{m+k}=T_mT_{k+1}+qT_{m-1}T_k$ be true, and show it's true for $n=k+1$.

$$T_{m+k+1}=pT_{m+k}+qT_{m+k-1}=p\left(T_mT_{k+1}+qT_{m-1}T_k\right)+q\left(T_mT_k+qT_{m-1}T_{k-1}\right)$$
$$=T_m\left(pT_{k+1}+qT_k\right)+qT_{m-1}\left(pT_k+qT_{k-1}\right)=T_mT_{k+2}+qT_{m-1}T_{k+1}$$
$$=T_mT_{k+1+1}+qT_{m-1}T_{k+1}.$$ Thus, the result is true for $n=k+1$.

1-Alternative proof of Theo (9),

$$T_m T_{n+1} + qT_{m-1}T_n = \left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) - (-\alpha\beta)\left(\frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}\right)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

$$= \frac{\alpha^{m+n+1}+\beta^{m+n+1}-\alpha^{n+1}\beta^m-\alpha^m\beta^{n+1}}{(\alpha-\beta)^2} + \frac{-\alpha^{m+n}\beta-\alpha^{m+n}\alpha+\alpha^m\beta^{n+1}+\alpha^{n+1}\beta^m}{(\alpha-\beta)^2}$$

$$= \frac{\alpha^{m+n+1}+\beta^{m+n+1}-\alpha^{m+n}\beta+\alpha\beta^{m+n}}{(\alpha-\beta)^2} = \frac{\alpha^{m+n}(\alpha-\beta)-\beta^{m+n}(\alpha-\beta)}{(\alpha-\beta)^2}$$

$$= \frac{\alpha^{m+n}+\beta^{m+n}}{\alpha-\beta} = T_{m+n}.$$

2- Alternative proof of Theo (9), (using matrix method)

We know that $\mathbf{U}^{m+n} = \mathbf{U}^m \times \mathbf{U}^n$ and, if

$$\mathbf{U} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}, \text{ then } \mathbf{U}^n = \begin{pmatrix} qT_{n-1} & T_n \\ qT_n & T_{n+1} \end{pmatrix}.$$

Then

$$\mathbf{U}^m \times \mathbf{U}^n = \mathbf{U}^{m+n} \Rightarrow \begin{pmatrix} qT_{m-1} & T_m \\ qT_m & T_{m+1} \end{pmatrix}\begin{pmatrix} qT_{n-1} & T_n \\ qT_n & T_{n+1} \end{pmatrix} = \begin{pmatrix} qT_{m+n-1} & T_{m+n} \\ qT_{m+n} & T_{m+n+1} \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} q^2T_{m-1}T_{n-1}+qT_mT_n & qT_{m-1}T_n+T_mT_{n+1} \\ q^2T_mT_{n-1}+qT_{m+1}T_n & qT_mT_n+T_{m+1}T_{n+1} \end{pmatrix} = \begin{pmatrix} qT_{m+n-1} & T_{m+n} \\ qT_{m+n} & T_{m+n+1} \end{pmatrix}$$

Equating the corresponding entries, then

$$q^2T_{m-1}T_{n-1}+qT_mT_n = qT_{m+n-1} \Rightarrow T_{m+n-1} = T_mT_n + qT_{m-1}T_{n-1}$$

$$q^2T_mT_{n-1}+qT_{m+1}T_n = qT_{m+n} \Rightarrow T_{m+n} = T_{m+1}T_n + qT_mT_{n-1}$$

$$qT_{m-1}T_n + T_mT_{n+1} = T_{m+n} \Rightarrow T_{m+n} = T_mT_{n+1} + qT_{m-1}T_n$$

$$qT_mT_n + T_{m+1}T_{n+1} = T_{m+n+1} \Rightarrow T_{m+n+1} = T_{m+1}T_{n+1} + qT_mT_n.$$

Note: Putting $n = n-m \geq 0$ then the above result can be written as

$$T_n = T_mT_{n-m+1} + qT_{m-1}T_{n-m}.$$

<u>Theo (9)</u>. The following relations are true

$(a)\quad T_{2n}=T_n\left(T_{n+1}+qT_{n-1}\right)$

$(b)\quad T_{2n+1}=qT_n^2+T_{n+1}^2$

$(c)\quad T_{m+n}=q^mT_n+(-1)^nT_m.$

Proof. (a)- We have

$$T_{2n}=\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}=\frac{\alpha^n-\beta^n}{\alpha-\beta}\left(\alpha^n+\beta^n\right)=T_n\left(\alpha^n+\beta^n\right)$$

Since $\left(\alpha^n+\beta^n\right)(\alpha-\beta)=\alpha^{n+1}-\beta^{n+1}-\alpha\beta\left(\alpha^{n-1}-\beta^{n-1}\right).$

Therefore

$$\alpha^n+\beta^n=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}-\alpha\beta\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)=T_{n+1}-\alpha\beta T_{n-1}.$$

Notice that $\alpha\beta=-q$.

(b)- For $n=0$, the result is true since $T_1=qT_0^2+T_1^2=1$.

Suppose the result is true for $n=k\geq 1$, i.e., let

$T_{2k+1}=qT_k^2+T_{k+1}^2$ is true, and show the result is true for $n=k+1$.

Consequently,

$$\begin{aligned}qT_{k+1}^2+T_{k+2}^2&=q\left(T_{2k+1}-qT_k^2\right)+T_{k+2}^2=T_{k+2}^2-q^2T_k^2+qT_{2k+1}\\&=\left(T_{k+2}-qT_k\right)\left(T_{k+2}+qT_k\right)+qT_{2k+1}=pT_{k+1}\left(T_{k+2}+qT_k\right)+qT_{2k+1}\\&=pT_{2k+2}+qT_{2k+1}=T_{2k+3}.\end{aligned}$$

This induction proof is completed.

(c)- For $n=0$, the result is true since $T_m=q^mT_0+(-1)^0T_m=T_m$.

Suppose the result is true for $n=k\geq 1$, i.e.,

let $T_{m+k}=q^mT_k+(-1)^kT_m$ is true, and show the result is true for $n=k+1$. Accordingly,

$$T_{k+m+1}-q^mT_{k+1}=pT_{k+m}+qT_{k+m-1}-q^m\left(pT_k+qT_{k-1}\right)$$
$$=\left(q-1\right)\left(T_{k+m}-q^mT_k\right)+q\left(T_{k+m-1}+q^mT_{k-1}\right)=\left(q-1\right)\left(-1\right)^kT_m+q\left(-1\right)^{k-1}T_m$$
$$=q\left(-1\right)^kT_m-\left(-1\right)^kT_m-q\left(-1\right)^kT_m=\left(-1\right)^{k+1}T_m.$$

Thus, the proof is completed.

Theo (10). (Cassini's identity). For any positive integers, p,q and n we have $T_{n+1}T_{n-1}-T_n^2=\left(-1\right)^nq^{n-1}$.

Proof. Let p,q and n be positive integers and $n\geq 2$.

Since $T_n=pT_{n-1}+qT_{n-2}$, $T_{n-1}=pT_{n-2}+qT_{n-3}$ and $T_{n+1}=pT_n+qT_{n-1}$.

Then

$$\begin{aligned}T_{n+1}T_{n-1}-T_n^2&=\left(pT_n+qT_{n-1}\right)T_{n-1}-T_n^2=pT_nT_{n-1}+qT_{n-1}^2-T_n^2\\&=qT_{n-1}^2+T_n\left(pT_{n-1}-T_n\right)=qT_{n-1}^2+T_n\left(-qT_{n-2}\right)\\&=qT_{n-1}^2-qT_nT_{n-2}=-q\left(T_nT_{n-2}-T_{n-1}^2\right)\end{aligned}$$

We can now repeat the above process on the last line to obtain

$$\begin{aligned}&=\left(-q\right)^2\left(T_{n-1}T_{n-3}-T_{n-2}^2\right)=\left(-q\right)^3\left(T_{n-2}T_{n-4}-T_{n-3}^2\right)=\cdots\\&=\left(-1\right)^nq^n\left(T_1T_{-1}-T_0^2\right)=\left(-1\right)^nq^{n-1},\end{aligned}$$

since $T_1=1$ and $T_{-1}=1/q$.

1-Alternative proof Theo (10). We have

$$T_{n+1}T_{n-1}-T_n^2=\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)-\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$$
$$=\frac{\alpha^{2n}+\beta^{2n}-\alpha^{n-1}\beta^{n+1}-\alpha^{n+1}\beta^{n-1}}{\left(\alpha-\beta\right)^2}-\frac{\alpha^{2n}-2\alpha^n\beta^n+\beta^{2n}}{\left(\alpha-\beta\right)^2}$$
$$=\frac{-\alpha^{n-1}+\beta^{n-1}-\alpha^{n-1}\beta^{n+1}+2\alpha^n\beta^n}{\left(\alpha-\beta\right)^2}$$

$$=\frac{\alpha^{n-1}\beta^{n-1}(-1)\left(\alpha^2-2\alpha\beta+\beta^2\right)}{(\alpha-\beta)^2}=\frac{(-1)^n q^{n-1}(\alpha-\beta)^2}{(\alpha-\beta)^2}=(-1)^n q^{n-1}.$$

2-**Alternative proof Theo (10) (**Using matrix method).

Since $\mathbf{U}=\begin{pmatrix}0 & 1\\ q & p\end{pmatrix}$, then $\det\mathbf{U}=-q$ and $\mathbf{U}^n=\begin{pmatrix}qT_{n-1} & T_n\\ qT_n & T_{n+1}\end{pmatrix}$, then taking determinants we have,

$$\det\mathbf{U}^n=\begin{vmatrix}qT_{n-1} & T_n\\ qT_n & T_{n+1}\end{vmatrix}=qT_{n-1}T_{n+1}-qT_n^2=q\left(T_{n+1}T_{n-1}-T_n^2\right)$$

$$=(\det\mathbf{U})^n=(-q)^n\Rightarrow T_{n+1}T_{n-1}-T_n^2=(-1)^n q^{n-1}.$$

<u>Theo (11)</u>.

For any nonnegative integer n we have

$$T_{n-r}T_{n+r}-T_n^2=(-1)^{n+1-r}T_r^2.$$

Proof. $T_{n-r}T_{n+r}-T_n^2=\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}.\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}-\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$

$$=\frac{\alpha^{2n}-\alpha^{n-r}\beta^{n+r}-\alpha^{n+r}\beta^{n-r}+\beta^{2n}-\alpha^{2n}+2\alpha^n\beta^n-\beta^{2n}}{(\alpha-\beta)^2}$$

$$=\frac{1}{(\alpha-\beta)^2}\left[-(\alpha\beta)^n\left(\frac{\beta}{\alpha}\right)^r-(\alpha\beta)^n\left(\frac{\alpha}{\beta}\right)^r+2(\alpha\beta)^n\right]$$

$$=\frac{(-1)^{n+1}}{(\alpha-\beta)^2}\left[\frac{\beta^{2r}+\alpha^{2r}}{(\alpha\beta)^r}-2\right]=(-1)^{n+1-r}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2=(-1)^{n+1-r}T_r^2.$$

<u>Theo (12)</u>. For any positive integers, p,q and n, we have

$$T_{n-2}T_{n+1}-T_{n-1}T_n=(-1)^{n-1}pq^{n-2}.$$

Proof. Let p,q and n are positive integers and $n \geq 2$. We get

$$T_{n-2}T_{n+1} - T_{n-1}T_n = \left(\frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta}\right)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right)\left(\frac{\alpha^{n} - \beta^{n}}{\alpha - \beta}\right)$$

$$= \frac{\alpha^{2n-1} + \beta^{2n-1} - \alpha^{n-2}\beta^{n+1} - \alpha^{n+1}\beta^{n-2}}{(\alpha - \beta)^2} - \frac{\alpha^{2n-1} - \alpha^{n}\beta^{n-1} - \alpha^{n-1}\beta^{n} + \beta^{2n-1}}{(\alpha - \beta)^2}$$

$$= \frac{-\alpha^{n-2} + \alpha^{n+1}\beta^{n-2} + \alpha^{n}\beta^{n-1} + \beta^{n+1}}{(\alpha - \beta)^2}$$

$$= \frac{\alpha^{n-2}\beta^{n-2}(-1)\left(\alpha^3 - \alpha^2\beta - \alpha\beta^2 + \beta^3\right)}{(\alpha - \beta)^2} = (-1)^{n-1} q^{n-2}(\alpha + \beta) = (-1)^{n-1} pq^{n-2}.$$

<u>Theo (13)</u>. For any positive integers, p,q, m, and n, $m \geq n$. Then

$$T_m^2 - T_{m-n}T_{m+n} = (-1)^{m-n} q^{m-n} T_n^2.$$

Proof. Let p,q, m, and n are positive integers, $m \geq n$. We have

$$T_m^2 - T_{m-n}T_{m+n} = \left(\frac{\alpha^{m} - \beta^{m}}{\alpha - \beta}\right)^2 - \left(\frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta}\right)\left(\frac{\alpha^{m+n} - \beta^{m+n}}{\alpha - \beta}\right)$$

$$= \frac{\alpha^{2m} + \beta^{2m} - 2\alpha^{m}\beta^{m} - \alpha^{n+1}\beta^{n-2}}{(\alpha - \beta)^2} - \frac{\alpha^{2m} - \alpha^{m+n}\beta^{m-n} - \alpha^{m-n}\beta^{m+n} + \beta^{2m}}{(\alpha - \beta)^2}$$

$$= \frac{-\alpha^{m-n}\beta^{m-n}\left(\alpha^{2n} - 2\alpha^{n}\beta^{n} + \beta^{2n}\right)}{(\alpha - \beta)^2} = \frac{(-1)^{m-n} q^{m-n}\left(\alpha^{n} - \beta^{n}\right)^2}{(\alpha - \beta)^2}$$

$$= (-1)^{m-n} q^{m-n} T_n^2.$$

<u>Theo (14).</u> For any positive integers, p,q, m, n and k. Then

$$T_{m+n}T_{m+k} - T_m T_{m+n+k} = (-q)^m T_n T_k.$$

Proof. Let p,q, m, n and k are positive integers. We have

$$T_{m+n}T_{m+k}-T_mT_{m+n+k}=\left(\frac{\alpha^{m+n}-\beta^{m+n}}{\alpha-\beta}\right)\left(\frac{\alpha^{m+k}-\beta^{m+k}}{\alpha-\beta}\right)$$

$$-\left(\frac{\alpha^{m}-\beta^{m}}{\alpha-\beta}\right)\left(\frac{\alpha^{m+n+k}-\beta^{m+n+k}}{\alpha-\beta}\right)$$

$$=\frac{\alpha^m\beta^m\left[-\alpha^n\beta^k-\alpha^k\beta^n+\alpha^{n+k}+\beta^{n+k}\right]}{(\alpha-\beta)^2}$$

$$=\frac{(-q)^m\left[-\beta^k\left(\alpha^n-\beta^n\right)+\alpha^k\left(\alpha^n-\beta^n\right)\right]}{(\alpha-\beta)^2}$$

$$=\frac{(-q)^m\left(\alpha^n-\beta^n\right)\left(\alpha^k-\beta^k\right)}{(\alpha-\beta)^2}=(-q)^mT_nT_k.$$

Theo (15). The following relations are true

$$\alpha^n=T_n\alpha+qT_{n-1}\ ,\ \beta^n=T_n\beta+qT_{n-1}\ \text{ and }\ T_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}.$$

Proof: (By mathematical induction)

if $n=1$ then $\alpha=T_1\alpha+qT_0$. So, $\alpha=\alpha$ as $T_0=0$, $T_1=1$. Therefor the result is true for $n=1$.

Also, for $n=2$ then

$$\alpha^2=\frac{p^2+2q+p\sqrt{p^2+4q}}{2}\ \text{ and }\ T_2\alpha+qT_1=\frac{p^2+2q+p\sqrt{p^2+4q}}{2}.$$

Thus, the result is true for $n=2$.

Suppose the result is true for $n=k$, i.e.,

let $\alpha^k=T_k\alpha+qT_{k-1}$ is true, we show the result is true for $n=k+1$.

Since $\alpha^2=T_2\alpha+qT_1$, $T_1=1$ and $T_2=\alpha$, then

$$\alpha^{k+1}=\alpha\alpha^k=\alpha\left(T_k\alpha+qT_{k-1}\right)=\alpha^2T_k+\alpha qT_{k-1}=\left(T_2\alpha+qT_1\right)T_k+\alpha qT_{k-1}$$

$$= \alpha T_2 T_k + qT_1 T_k + \alpha q T_{k-1} = \alpha\left(\alpha T_k + qT_{k-1}\right) + qT_k = \alpha T_{k+1} + qT_k .$$

Thus, the result is true for $n = k+1$. So, we can say that,

$$\alpha^n = T_n \alpha + qT_{n-1} \text{ for all } n \in \mathbb{N}.$$

Similarly, we can prove $\beta^n = T_n \beta + qT_{n-1}$.

Subtracting $\beta^n = T_n \beta + qT_{n-1}$ from $\alpha^n = T_n \alpha + qT_{n-1}$ and dividing the result by $\alpha - \beta$, we obtain

$$\alpha^n - \beta^n = T_n \alpha + qT_{n-1} - \left(T_n \beta + qT_{n-1}\right) = \left(\alpha - \beta\right) T_n \Rightarrow T_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

<u>**Theo (16)**</u>. The relation $\lim_{n \to \infty} \frac{T_{n+1}}{T_n} = \alpha$ is true

Proof. We have

$$\frac{T_{n+1}}{T_n} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \frac{\alpha\left(\alpha^n - \beta^n\right) + \beta^n\left(\alpha - \beta\right)}{\alpha^n - \beta^n} = \alpha + \frac{\alpha - \beta}{\left(\frac{\alpha}{\beta}\right)^n - 1}.$$

Since $\alpha > \beta$, then $\frac{\alpha - \beta}{\left(\frac{\alpha}{\beta}\right)^n - 1} \to 0$ as $n \to \infty$, implies the result.

<u>**Theo (17)**</u>. The following relation is true

$$T_n - c^{n-1} = \left(p - c\right) T_{n-1} + \left[\left(p - c\right)c + q\right]\left[T_0 c^{n-2} + T_1 c^{n-3} + \ldots + T_{n-2}\right],$$

where $1 \le c \le p$.

Proof: (By mathematical induction).

If $n = 2$ then

$$T_2 - c = \left(p - c\right) T_1 + \left[\left(p - c\right)c + q\right] T_0 ,$$

since $T_2 = pT_1 + qT_0 = pT_1$, then

$$pT_1 + qT_0 - c = (p-c)T_1 \Rightarrow (p-c)T_1 = (p-c)T_1.$$

Thus, the result is true for $n=2$.

Suppose the result is true for $n=k$, i.e., let

$T_k - c^{k-1} = (p-c)T_{k-1} + \left[(p-c)c+q\right]\left[T_0c^{k-2} + T_1c^{k-3} + ... + T_{k-2}\right]$ is true.

We show the result is true for $n=k+1$.

$$\begin{aligned}
&(p-c)T_k + \left[(p-c)c+q\right]\left[T_0c^{k-1} + T_1c^{k-2} + ... + T_{k-1}\right] \\
&= (p-c)T_k + \left[(p-c)c+q\right]\left[T_{k-1} + cT_{k-2} + c^2T_{k-3} + \cdots + c^{k-2}T_1 + c^{k-1}T_0\right] \\
&= (pT_k + qT_{k-1}) - cT_k + q\left[cT_{k-2} + c^2T_{k-3} + \cdots + c^{k-2}T_1 + c^{k-1}T_0\right] \\
&\qquad + p\left[cT_{k-1} + c^2T_{k-2} + \cdots + c^{k-1}T_1 + c^kT_0\right] \\
&\qquad + c\left[cT_{k-1} + c^2T_{k-2} + \cdots + c^{k-1}T_1 + c^kT_0\right] \\
&= T_{k+1} - cT_k + c(pT_{k-1} + qT_{k-2}) + c^2(pT_{k-2} + qT_{k-3}) + \cdots + c^{k-1}(pT_1 + qT_0) \\
&\qquad - c^2T_{k-1} - c^3T_{k-2} - \cdots - c^{k-1}T_2 - c^kT_1 - c^{k+1}T_0 \\
&= T_{k+1} - cT_k + cT_k + c^2T_{k-1} - c^2T_{k-1} + \cdots + c^{k-1}T_2 - c^2T_{k-1} - \cdots - c^{k-1}T_2 - c^k
\end{aligned}$$

$= T_{k+1} - c^k$. Thus, the result is true for $n=k+1$.

<u>Note that</u> if $c=p$ in above result then we have,

$$T_n - p^{n-1} = q\left[T_0p^{n-2} + T_1p^{n-3} + ... + T_{n-2}\right] \Rightarrow T_n = p^{n-1} + q\sum_{i=1}^{n-2} p^{i-1}T_{n-1-i}.$$

Exercises.

Solve the following Questions:

In the Fibonacci sequence $\{T_n = T_n(a,b;p,q)\}$: prove that

(1)- The sum of the first n terms given by

$$\sum_{i=0}^{n-1} T_i = \frac{T_n + qT_{n-1} + pa - a - b}{p+q-1},$$

where $p+q \neq 1$ and $p,q \geq 0$.

(2)- The explicit formula for the Fibonacci sequence $\{T_n = T_n(a,b;p,q)\}$ is given as

$$T_n = a\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k}{k} p^{n-2k} q^k + \left(\frac{b}{p} - a\right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} p^{n-2k} q^k,$$

where $a,b,p,q \in Z$ and $n \geq 1$.

(3)- Find few terms of in the following special cases of the Fibonacci sequence $\{T_n = T_n(a,b;p,q)\}$:

i- For $a = 0,\ b = 1$

ii- For $a = p = 2,\ q = 1$ and $b = 3$,

iii- For $a = 0,\ b = 1$, and $q = 1$.

(4)- The generalized sequences $\{T_n = T_n(a,b;p,q)\}$ have several famous number sequences as special cases. In the following table, for different values of a, b, p & q many sequences can be determined. Deduce these formulae?

Table 1: Some famous sequences

Name	$T_n(a,b;p,q)$	**Characteristic polynomial**	**Generating function**
Fibonacci	$\mathrm{F}_n(0,0;1,1)$	$1+x-x^2=0$	$G(x)=\frac{x}{1-x-x^2}$
Pell	$\mathrm{P}_n(0,1;2,1)$	$1+2x-x^2=0$	$G(x)=\frac{x}{1-2x-x^2}$
Jacobsthal	$\mathrm{J}_n(0,1;1,2)$	$2+x-x^2=0$	$G(x)=\frac{x}{1-x-2x^2}$
Mersenne	$M_n(0,1;3,-2)$	$2-3x+x^2=0$	$G(x)=\frac{x}{1-3x+2x^2}$
Lucas	$\mathrm{L}_n(2,1;1,1)$	$1+x-x^2=0$	$G(x)=\frac{2-x}{1-x-x^2}$
P-Lucas	$p_n(2,2;2,1)$	$1+2x-x^2=0$	$G(x)=\frac{2-2x}{1-2x-x^2}$
J-Lucas	$J_n(2,1;1,2)$	$2+x-x^2=0$	$G(x)=\frac{2-x}{1-x-2x^2}$
M-Lucas	$m_n(2,3;3,-2)$	$2-3x+x^2=0$	$G(x)=\frac{2-3x}{1-3x+2x^2}$