EXAMPLE 1. For an equilateral triangle, $k_1 = k_2 = k_3 = 2/3$. It is convenient to write $x_1 = -1$, $x_2 = 1$, and $x_3 = \infty$ and to use equation (2), with $z_0 = 1$, $A = 1$, and $B = 0$. The transformation then becomes

$$w = \int_1^z (s+1)^{-2/3}(s-1)^{-2/3}\,ds. \tag{3}$$

The image of the point $z = 1$ is clearly $w = 0$; that is, $w_2 = 0$. If $z = -1$ in this integral, one can write $s = x$, where $-1 < x < 1$. Then

$$x + 1 > 0 \quad \text{and} \quad \arg(x+1) = 0,$$

while

$$|x - 1| = 1 - x \quad \text{and} \quad \arg(x-1) = \pi.$$

Hence

$$\begin{aligned} w &= \int_1^{-1} (x+1)^{-2/3}(1-x)^{-2/3}\exp\left(-\frac{2\pi i}{3}\right)dx \\ &= \exp\left(\frac{\pi i}{3}\right)\int_0^1 \frac{2\,dx}{(1-x^2)^{2/3}} \end{aligned} \tag{4}$$

when $z = -1$. With the substitution $x = \sqrt{t}$, the last integral here reduces to a special case of the one used in defining the beta function (Exercise 5, Sec. 91). Let b denote its value, which is positive:

$$b = \int_0^1 \frac{2\,dx}{(1-x^2)^{2/3}} = \int_0^1 t^{-1/2}(1-t)^{-2/3}\,dt = B\left(\frac{1}{2}, \frac{1}{3}\right). \tag{5}$$

The vertex w_1 is, therefore, the point (Fig. 182)

$$w_1 = b\exp\frac{\pi i}{3}. \tag{6}$$

The vertex w_3 is on the positive u axis because

$$w_3 = \int_1^\infty (x+1)^{-2/3}(x-1)^{-2/3}\,dx = \int_1^\infty \frac{dx}{(x^2-1)^{2/3}}.$$

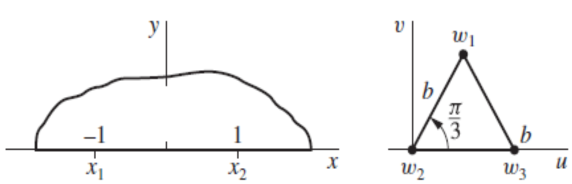

FIGURE 182

But the value of w_3 is also represented by integral (3) when z tends to infinity along the negative x axis; that is,

$$w_3 = \int_1^{-1} (|x+1||x-1|)^{-2/3} \exp\left(-\frac{2\pi i}{3}\right) dx$$
$$+ \int_{-1}^{-\infty} (|x+1||x-1|)^{-2/3} \exp\left(-\frac{4\pi i}{3}\right) dx.$$

In view of the first of expressions (4) for w_1, then,

$$w_3 = w_1 + \exp\left(-\frac{4\pi i}{3}\right) \int_{-1}^{-\infty} (|x+1||x-1|)^{-2/3} dx$$
$$= b \exp\frac{\pi i}{3} + \exp\left(-\frac{\pi i}{3}\right) \int_1^{\infty} \frac{dx}{(x^2-1)^{2/3}},$$

or

$$w_3 = b \exp\frac{\pi i}{3} + w_3 \exp\left(-\frac{\pi i}{3}\right).$$

Solving for w_3, we find that

$$w_3 = b. \tag{7}$$

We have thus verified that the image of the x axis is the equilateral triangle of side b shown in Fig. 182. We can also see that

$$w = \frac{b}{2} \exp\frac{\pi i}{3} \quad \text{when} \quad z = 0.$$

When the polygon is a rectangle, each $k_j = 1/2$. If we choose ± 1 and $\pm a$ as the points x_j whose images are the vertices and write

$$g(z) = (z+a)^{-1/2}(z+1)^{-1/2}(z-1)^{-1/2}(z-a)^{-1/2}, \tag{8}$$

where $0 \le \arg(z - x_j) \le \pi$, the Schwarz–Christoffel transformation becomes

$$w = -\int_0^z g(s)\, ds, \tag{9}$$

except for a transformation $W = Aw + B$ to adjust the size and position of the rectangle. Integral (9) is a constant times the elliptic integral

$$\int_0^z (1-s^2)^{-1/2}(1-k^2 s^2)^{-1/2}\, ds \qquad \left(k = \frac{1}{a}\right),$$

but the form (8) of the integrand indicates more clearly the appropriate branches of the power functions involved.

EXAMPLE 2. Let us locate the vertices of the rectangle when $a > 1$. As shown in Fig. 183, $x_1 = -a$, $x_2 = -1$, $x_3 = 1$, and $x_4 = a$. All four vertices can be described in terms of two positive numbers b and c that depend on the value of a in the following manner:

$$b = \int_0^1 |g(x)|\,dx = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(a^2-x^2)}}, \tag{10}$$

$$c = \int_1^a |g(x)|\,dx = \int_1^a \frac{dx}{\sqrt{(x^2-1)(a^2-x^2)}}. \tag{11}$$

If $-1 < x < 0$, then

$$\arg(x+a) = \arg(x+1) = 0 \qquad \text{and} \qquad \arg(x-1) = \arg(x-a) = \pi;$$

hence

$$g(x) = \left[\exp\left(-\frac{\pi i}{2}\right)\right]^2 |g(x)| = -|g(x)|.$$

If $-a < x < -1$, then

$$g(x) = \left[\exp\left(-\frac{\pi i}{2}\right)\right]^3 |g(x)| = i|g(x)|.$$

Thus

$$\begin{aligned} w_1 &= -\int_0^{-a} g(x)\,dx = -\int_0^{-1} g(x)\,dx - \int_{-1}^{-a} g(x)\,dx \\ &= \int_0^{-1} |g(x)|\,dx - i\int_{-1}^{-a} |g(x)|\,dx = -b + ic. \end{aligned}$$

It is left to the exercises to show that

$$w_2 = -b, \qquad w_3 = b, \qquad w_4 = b + ic. \tag{12}$$

The position and dimensions of the rectangle are shown in Fig. 183.

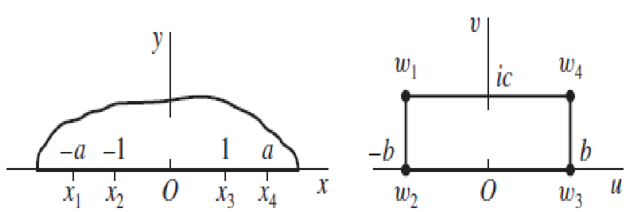

FIGURE 183

EXAMPLE 1. Let us map the half plane $y \geq 0$ onto the semi-infinite strip

$$-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad v \geq 0.$$

We consider the strip as the limiting form of a triangle with vertices w_1, w_2, and w_3 (Fig. 184) as the imaginary part of w_3 tends to infinity.

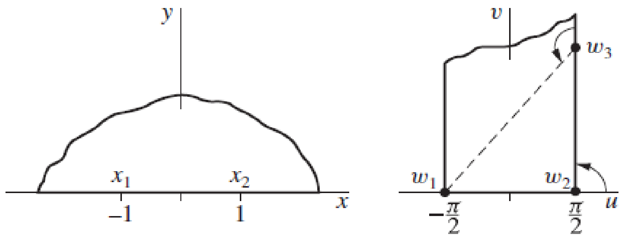

FIGURE 184

The limiting values of the exterior angles are

$$k_1\pi = k_2\pi = \frac{\pi}{2} \quad \text{and} \quad k_3\pi = \pi.$$

We choose the points $x_1 = -1$, $x_2 = 1$, and $x_3 = \infty$ as the points whose images are the vertices. Then the derivative of the mapping function can be written

$$\frac{dw}{dz} = A(z+1)^{-1/2}(z-1)^{-1/2} = A'(1-z^2)^{-1/2}.$$

Hence $w = A' \sin^{-1} z + B$. If we write $A' = 1/a$ and $B = b/a$, it follows that

$$z = \sin(aw - b).$$

This transformation from the w to the z plane satisfies the conditions $z = -1$ when $w = -\pi/2$ and $z = 1$ when $w = \pi/2$ if $a = 1$ and $b = 0$. The resulting transformation is

مناقشة تمارين

Schwarz–Christoffel Transformation

From 1-5

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