LECTURE 6

UNIT 5

KARNAUGH MAPS

OBJECTIVES

1. Given a function (completely or incompletely specified) of three to five variables, plot it on a karnaugh map. The function may be given in **minterm, maxterm, or algebraic form**.

2. Determine the essential prime implicants of a function from a map.

3. Obtain the minimum sum-of-products or minimum product-of-sums form of a function from the map.

4. Determine all of the prime implicants of a function from a map.

5. Understand the relation between operations performed using the map and the corresponding algebraic operations.

KARNAUGH MAPS

Switching functions can generally be simplified by using the algebraic techniques

described in Unit 3. However, two problems arise when algebraic procedures are used:

1. The procedures are difficult to apply in a systematic way.

2. It is difficult to tell when you have arrived at a minimum solution.

The Karnaugh map method studied in this unit and the Quine-McCluskey procedure studied in Unit 6 overcome these difficulties by providing systematic methods for simplifying switching functions.

5.1 MINIMUM FORMS OF SWITCHING FUNCTIONS

- When a function is realized using AND and OR gates, the cost of realizing the function is directly related to the number of gates and gate inputs used.
- **The karnaugh map techniques developed in this unit lead directly to minimum cost two-level circuits composed of AND and OR gates.**
- An expression consisting of a sum of product terms corresponds directly to a two-level circuit composed of a group of AND gates feeding a single OR gate (see figure 2-5).
- Similarly, a product-of sums expression corresponds to a two-level circuit composed of OR gates feeding a single AND gate (see figure 2-6).
- **Therefore, to find minimum cost two-level AND-OR gate circuits, we must find minimum expressions in sum-of-products or product-of-sums form.**

5.1 MINIMUM FORMS OF SWITCHING FUNCTIONS

- **A minimum sum-of-products expression** for a function is defined as a sum of product terms which (a) has a minimum number of terms and (b) of all those expressions which have the same minimum number of terms, has a minimum number of literals.
- The minimum sum of products corresponds directly to a minimum two-level gate circuit which has (a) a minimum number of gates and (b) a minimum Number of gate inputs.
- Unlike the minterm expansion for a function, the minimum sum of products is not necessarily unique; that is, a given function may have two different minimum sum-of-products forms, each with the same number of terms and the same number of literals.

5.1 MINIMUM FORMS OF SWITCHING FUNCTIONS

- Given a minterm expansion, the minimum sum-of products form can often be obtained by the following procedure:

1. Combine terms by using **XY'+XY = X**. Do this repeatedly to eliminate as many literals as possible. A given term may be used more than once becaus

   X + X =X.

2. Eliminate redundant terms by using the consensus theorem or other theorems.

5.1 MINIMUM FORMS OF SWITCHING FUNCTIONS

EXAMPLE: Find a minimum sum-of-products expression for

$$F(a, b, c) = \Sigma\, m\,(0, 1, 2, 5, 6, 7)$$

$$F = a'b'c' + a'b'c + a'bc' + ab'c + abc' + abc$$

$$= a'b' + b'c + bc' + ab \qquad (5\text{-}1)$$

None of the terms in the above expression can be eliminated by consensus. However, combining terms in a different way leads directly to a minimum sum of products:

$$F = a'b'c' + a'b'c + a'bc' + ab'c + abc' + abc$$

$$= a'b' + bc' + ac \qquad (5\text{-}2)$$

5.1 MINIMUM FORMS OF SWITCHING FUNCTIONS

- **A minimum product-of-sums expression** for a function is defined as a product of sum terms which (a) has a minimum number of factors, and (b) of all those expressions which have the same number of factors, has a minimum number of literals.
- Unlike the maxterm expansion, the minimum product-of-sums form of a function is not necessarily unique.
- Given a maxterm expansion, the minimum product of sums can often be obtained by a procedure similar to that used in the minimum sum-of-products case, except that the theorem is used to combine terms.

5.1 MINIMUM FORMS OF SWITCHING FUNCTIONS

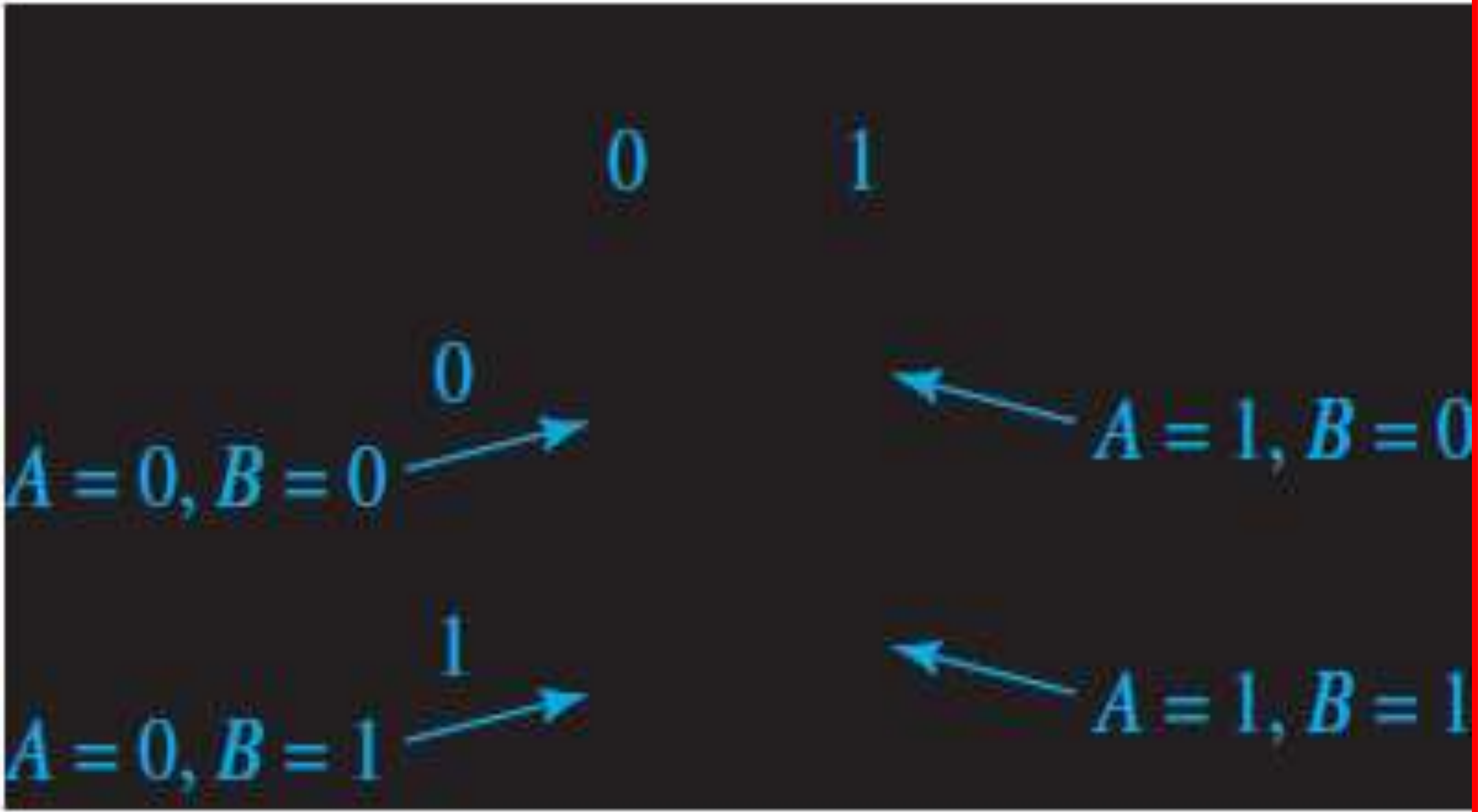
- **Example**

$$
\begin{aligned}
&(A + B' + C + D')(A + B' + C' + D')(A + B' + C' + D)(A' + B' + C' + D)(A + B + C' + D)(A' + B + C' + D) \\
&= (A + B' + D') \quad (A + B' + C') \quad (B' + C' + D) \quad (B + C' + D) \\
&= (A + B' + D') \quad \underbrace{(A + B' + C')} \quad (C' + D) \\
&= (A + B' + D')(C' + D) \qquad (5\text{-}3)
\end{aligned}
$$

eliminate by consensus

5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- Just like a truth table, the karnaugh map of a function specifies the value of the function for every combination of values of the independent variables. A two-variable karnaugh map is shown. The values of one variable are listed across the top of the map, and the values of the other variable are listed on the left side. Each square of the map corresponds to a pair of values for *A* and *B* as indicated.


0
1
0
$A = 0, B = 0$
$A = 1, B = 0$
1
$A = 0, B = 1$
$A = 1, B = 1$

5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

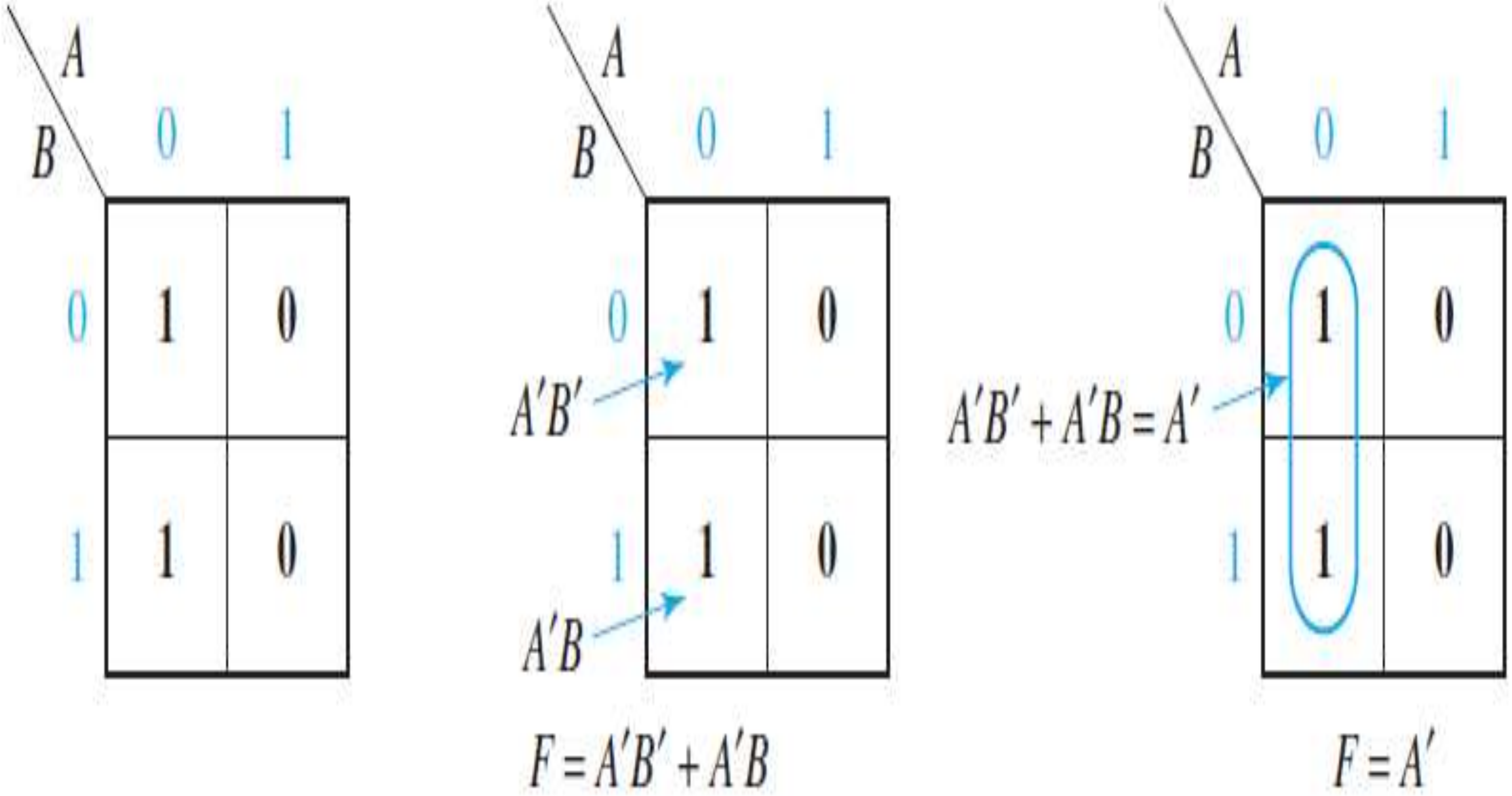
- Figure 5-1 shows the truth table for a function F and the corresponding Karnaugh map.
- **Note that** the value of F for A = B = 0 is plotted in the upper left square, and the other map entries are plotted in a similar way in Figure 5-1(b).
- Each 1 on the map corresponds to a minterm of F.
- We can read the minterms from the map just like we can read them from the truth table.

5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- A 1 in square 00 of Figure 5-1(c) indicates that A’B’ is a minterm of F.
- Similarly, a 1 in square 01 indicates that A’B is a minterm.
- Minterms in adjacent squares of the map can be combined since they differ in only one variable.
- Thus, A’B’ and A’B combine to form A’, and this is indicated by looping the corresponding 1’s on the map in Figure 5-1(d).

A	B	F
0	0	1
0	1	1
1	0	0
1	1	0

(a)



(b) (c) (d)

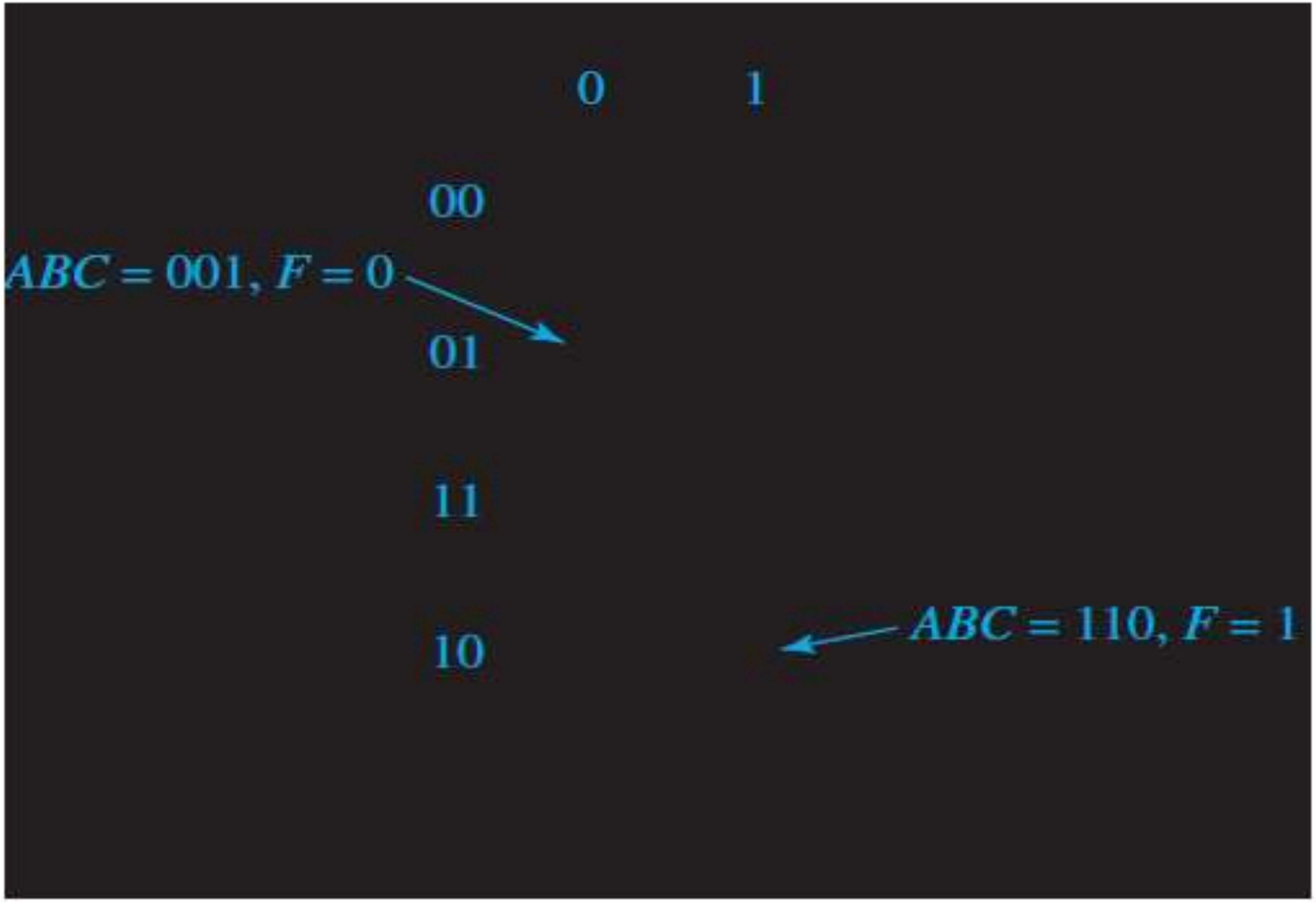
5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- Figure 5-2 shows a three-variable truth table and the corresponding karnaugh map.
- The value of one variable (A) is listed across the top of the map, and the values of the other two variables (B, C) are listed along the side of the map.
- The rows are labeled in the sequence 00, 01, 11, 10 so that values in adjacent rows differ in only one variable.
- For each combination of values of the variables, the value of f is read from the truth table and plotted in the appropriate map square.

FIGURE 5-2: Truth Table and Karnaugh Map for Three-Variable Function

A B C	F
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	1
1 0 1	0
1 1 0	1
1 1 1	0

(a)



5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

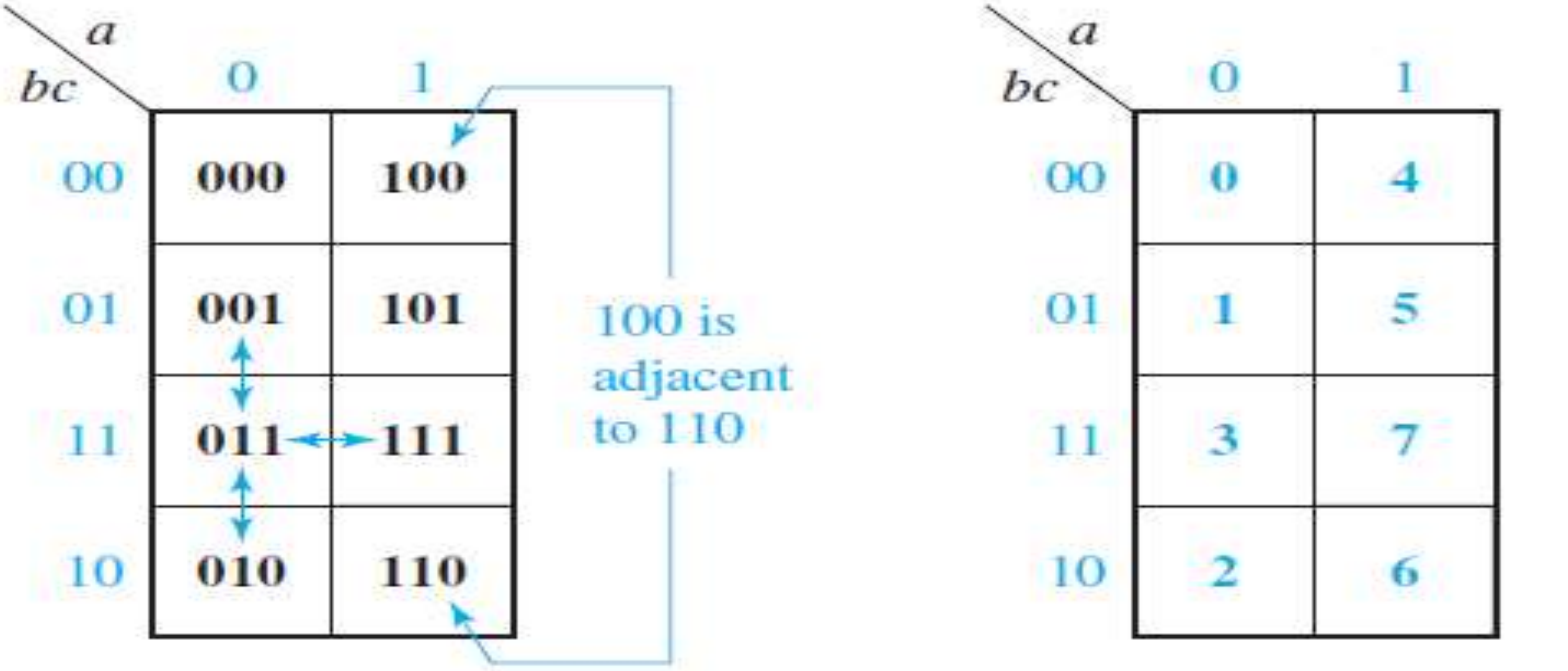
- **For example,**

for the input combination $ABC = 001$, the value $F = 0$ is plotted in the square for which $A = 0$ and $BC = 01$. For the combination $ABC = 110$, $F = 1$ is plotted in the $A = 1$, $BC = 10$ square.

5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- Figure 5-3 shows the location of the minterms on a three-variable map.
- Minterms in adjacent squares of the map differ in only one variable and therefore can be combined using the theorem $XY' = XY + X$.
- For example, minterm 011 ($a'bc$) is adjacent to the three minterms with which it can be combined—001 ($a'b'c$), 010 ($a'bc'$), and 111 (abc).
- In addition to squares which are physically adjacent, the top and bottom rows of the map are defined to be adjacent because the corresponding minterms in these rows differ in only one variable.
- Thus 000 and 010 are adjacent, and so are 100 and 110.

FIGURE 5-3: Location of Minterms on a Three-Variable Karnaugh Map



(a) Binary notation

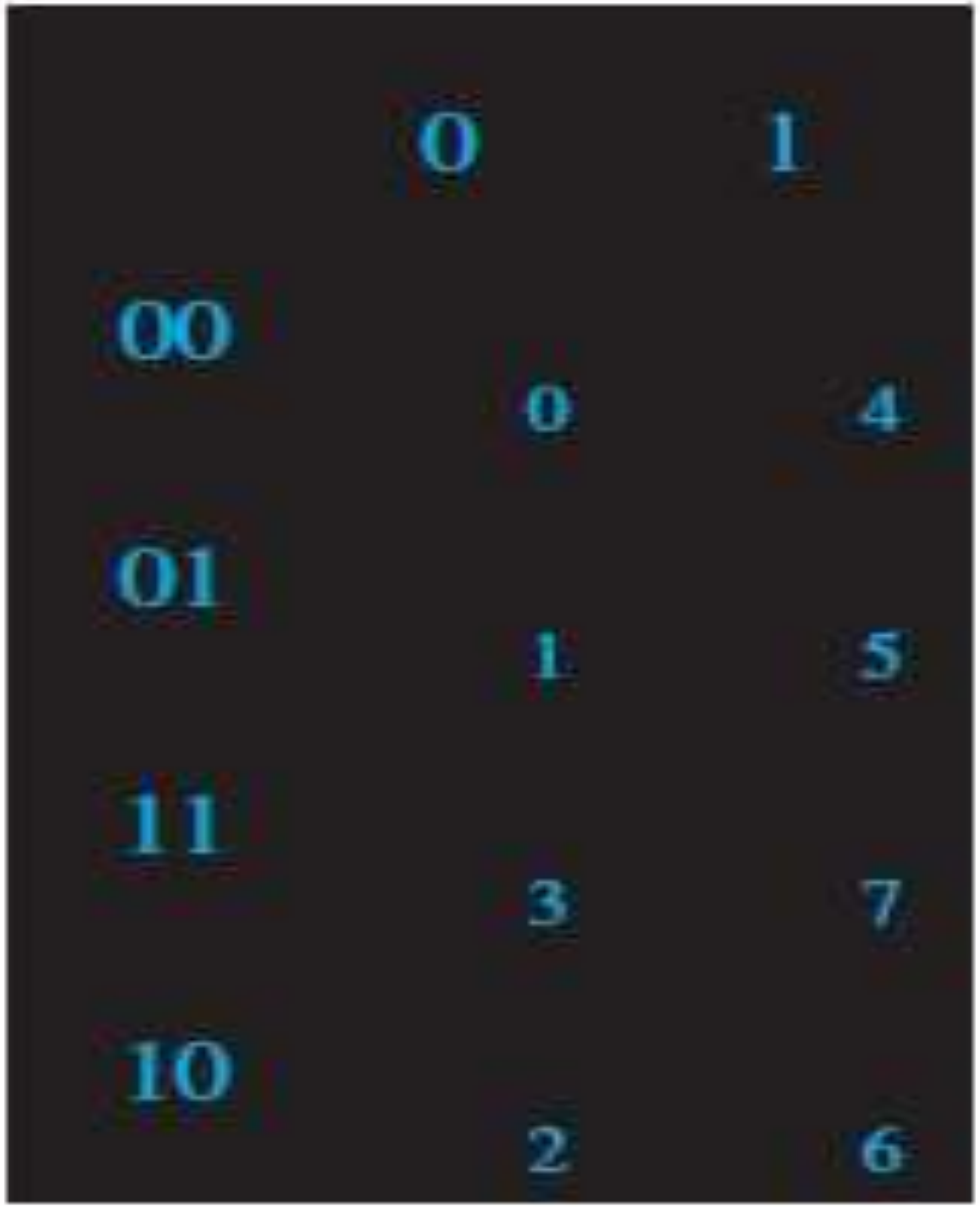
(b) Decimal notation

5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- Given the minterm expansion of a function, it can be plotted on a map by placing 1's in the squares which correspond to minterms of the function and 0's in the remaining squares (the 0's may be omitted if desired).
- Figure 5-4 shows the plot of $F(a, b, c) = m_1 + m_3 + m_5$.
- If F is given as a maxterm expansion, the map is plotted by placing 0's in the squares which correspond to the maxterms and then by filling in the remaining squares with 1's. Thus,

$F(a, b, c) = M_0 M_2 M_4 M_6 M_7$ gives the same map as figure 5-4.

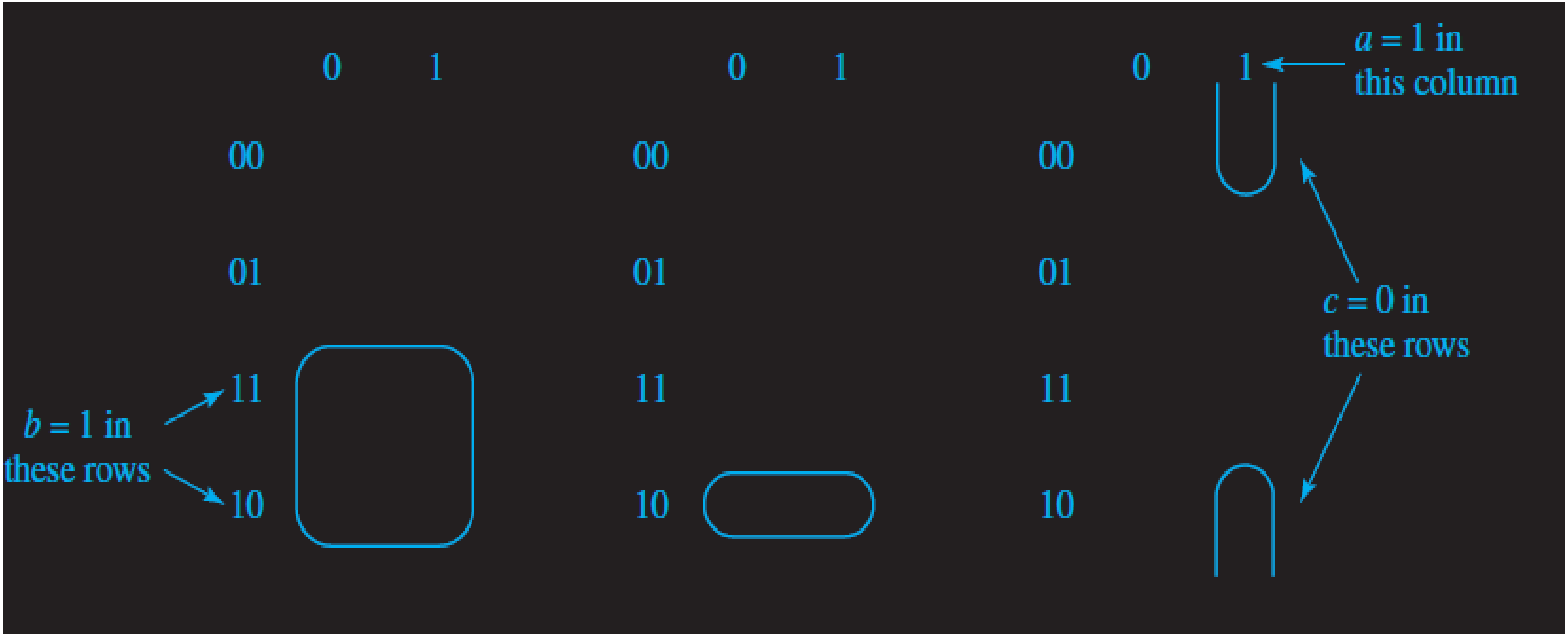
FIGURE 5-4
Karnaugh Map of $F(a, b, c) = \Sigma\, m(1, 3, 5) = \Pi\, M(0, 2, 4, 6, 7)$



5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- Figure 5-5 illustrates how product terms can be plotted on Karnaugh maps. To plot the term b, 1's are entered in the four squares of the map where b = 1. The term bc' is 1 when b = 1 and c = 0, so 1's are entered in the two squares in the bc = 10 row. The term ac' is 1 when a = 1 and c = 0, so 1's are entered in the a = 1 column in the rows where c = 0.

FIGURE 5-5: Karnaugh Maps for Product Terms

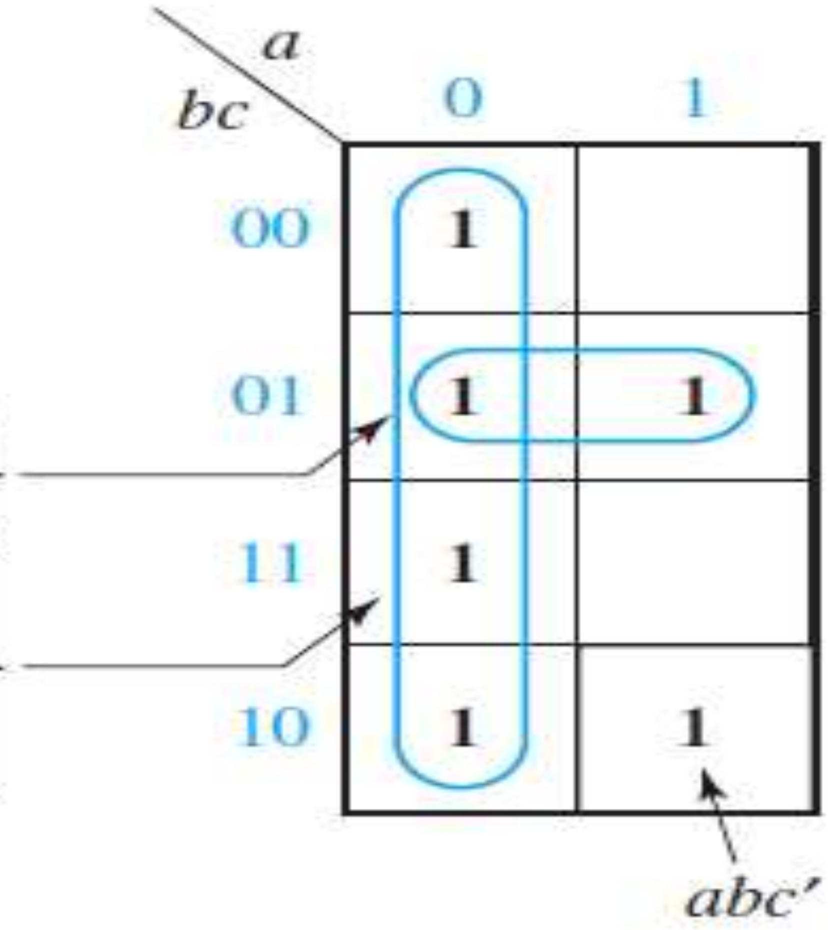


For example, given that

$$f(a, b, c) = abc' + b'c + a'$$

we would plot the map as follows:

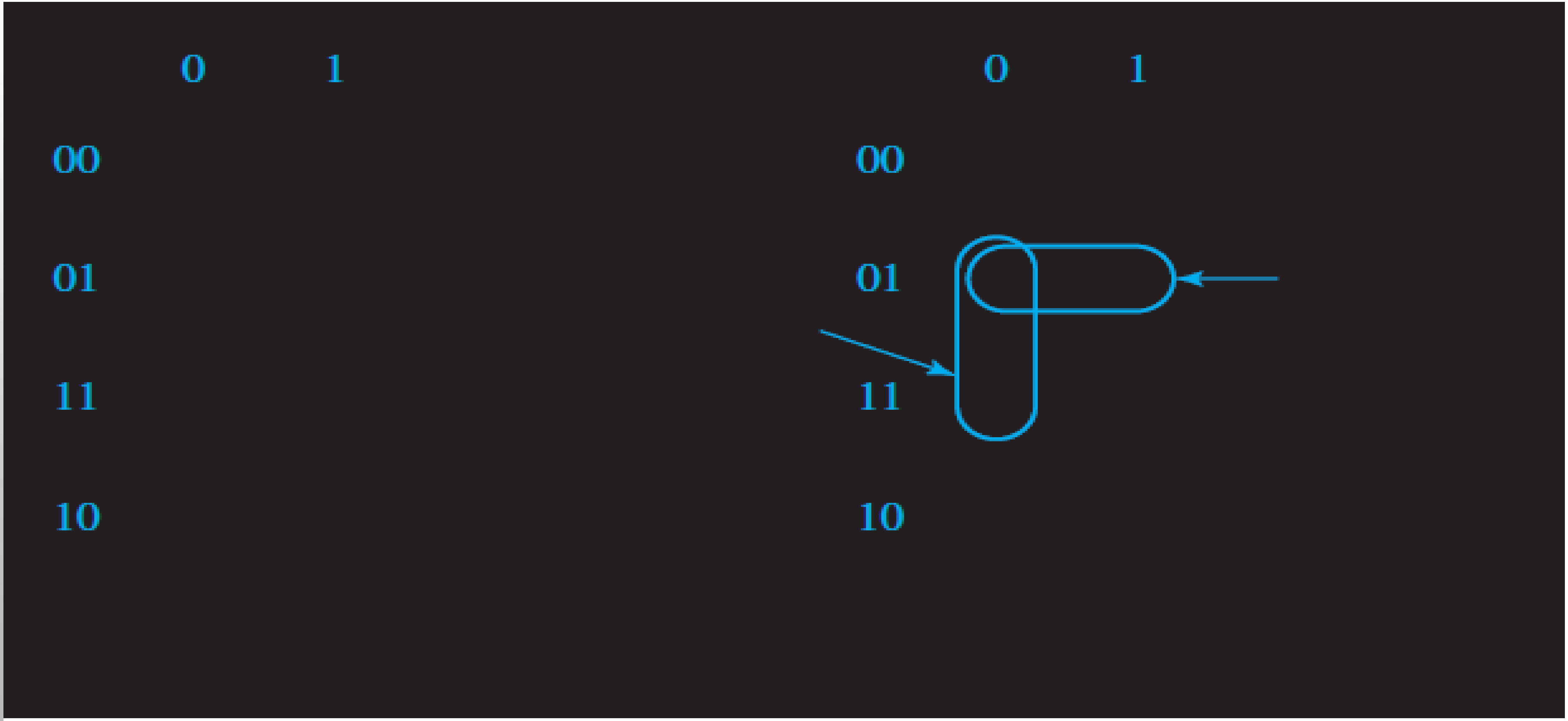
1. The term abc' is 1 when $a = 1$ and $bc = 10$, so we place a 1 in the square which corresponds to the $a = 1$ column and the $bc = 10$ row of the map.
2. The term $b'c$ is 1 when $bc = 01$, so we place 1's in both squares of the $bc = 01$ row of the map.
3. The term a' is 1 when $a = 0$, so we place 1's in all the squares of the $a = 0$ column of the map. (Note: Since there already is a 1 in the $abc = 001$ square, we do not have to place a second 1 there because $x + x = x$.)



5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- Figure 5-6 illustrates how a simplified expression for a function can be derived using a Karnaugh map.
- The function to be simplified is first plotted on a Karnaugh map in Figure 5-6(a).
- **Terms in adjacent squares on the map differ in only one variable and can be combined using the theorem XY' + XY = X**
- **Thus a'b'c and a'bc combine to form ac, and a'b'c and ab'c combine to form bc, as shown in Figure 5-6(b).**
- الذي يتغير يختفي

FIGURE 5-6: Simplification of a Three-Variable Function



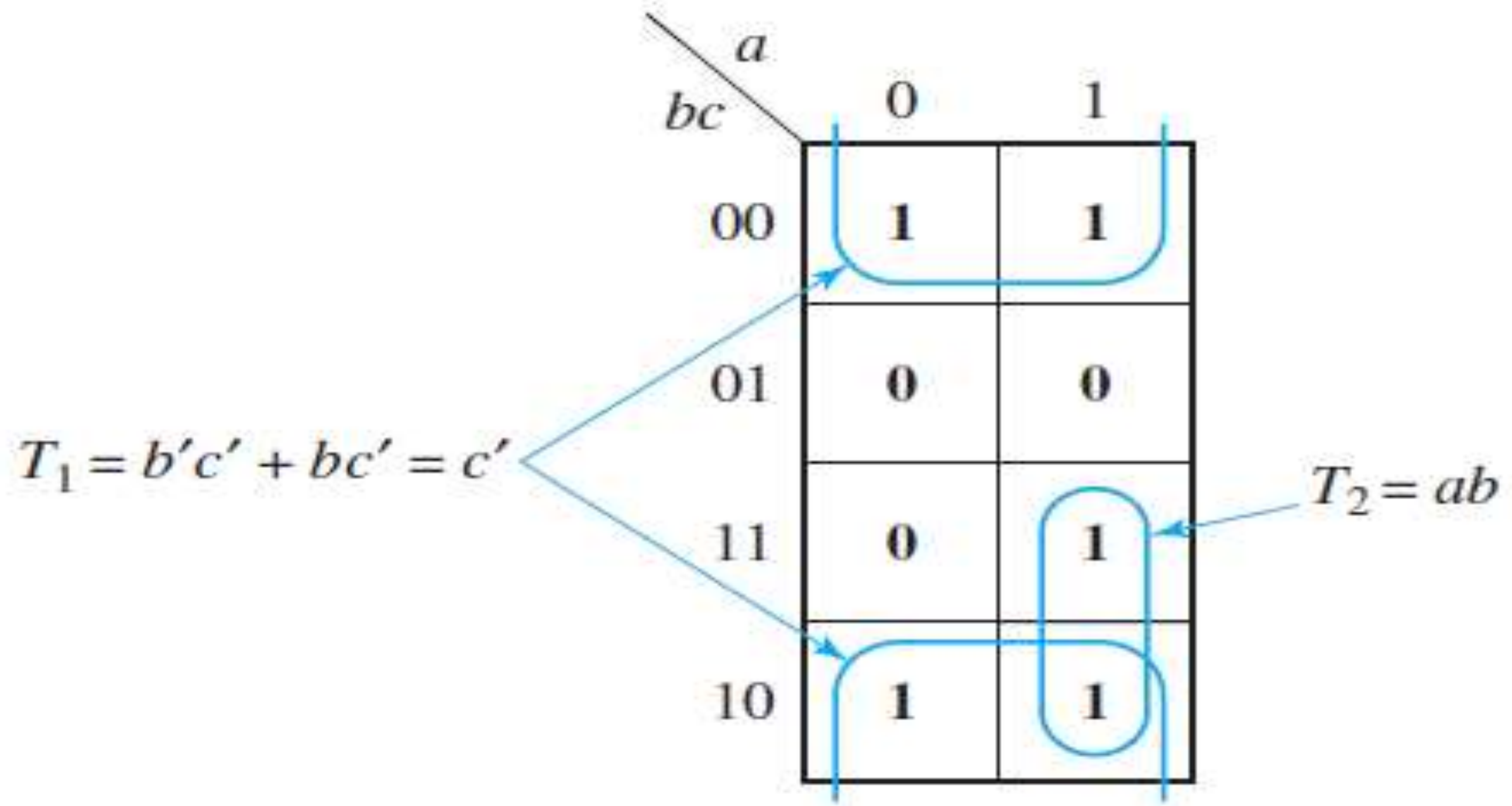
5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- Figure 5-6 illustrates how a simplified expression for a function can be derived using a Karnaugh map.
- The function to be simplified is first plotted on a Karnaugh map in Figure 5-6(a).
- **Terms in adjacent squares on the map differ in only one variable and can be combined using the theorem XY' + XY = X**
- **Thus a'b'c and a'bc combine to form ac, and a'b'c and ab'c combine to form bc, as shown in Figure 5-6(b).**
- الذي يتغير يختفي

5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

The map for the complement of F (Figure 5-7) is formed by replacing 0's with 1's and 1's with 0's on the map of F. To simplify F', note that the terms in the top row combine to form $b'c'$, and the terms in the bottom row combine to form bc'. Because $b'c'$ and bc' differ in only one variable, the top and bottom rows can then be combined to form a group of four 1's, thus eliminating two variables and leaving $T_1 = c'$. The remaining 1 combines, as shown, to form $T_2 = ab$, so the minimum sum-of-products form for F' is $c' + ab$.

FIGURE 5-7: Complement of Map in Figure 5.6(a)

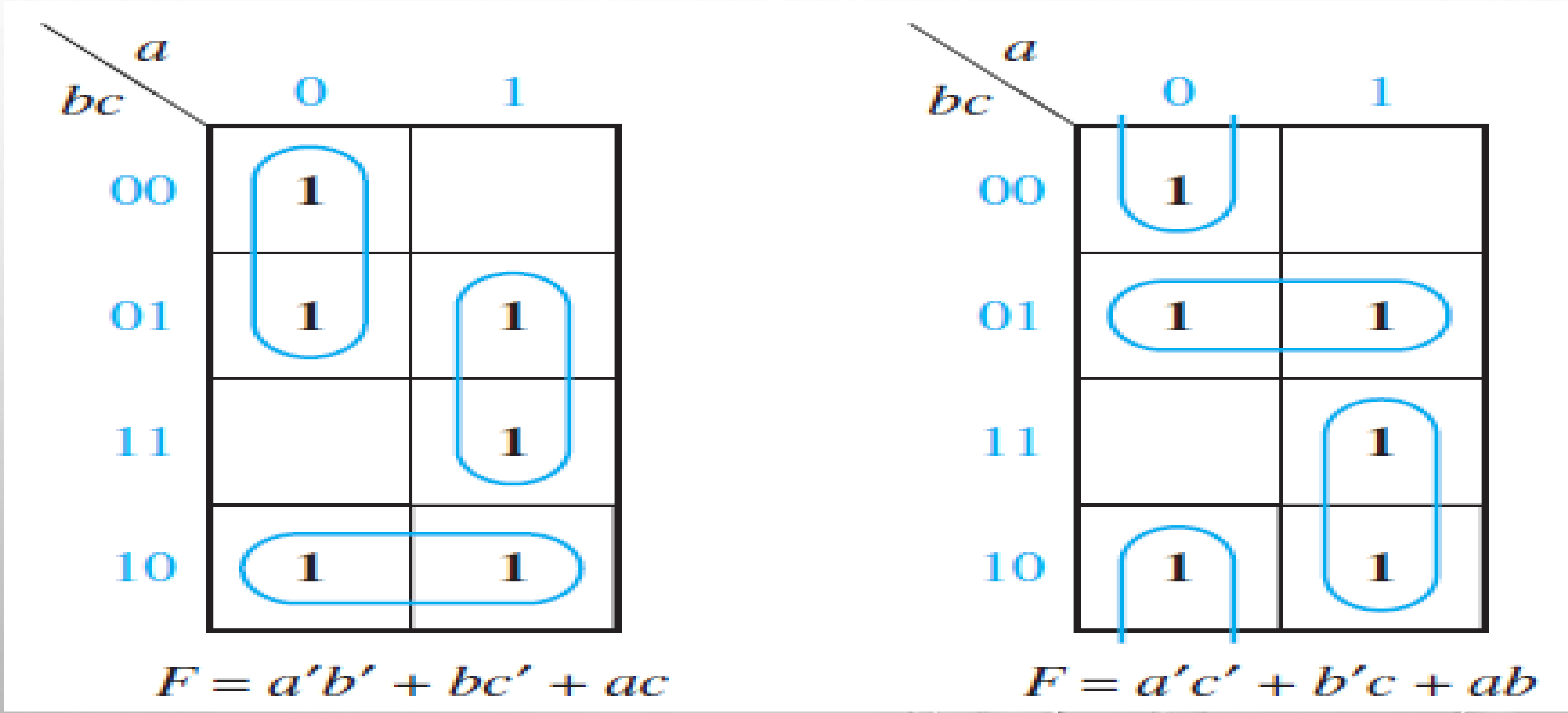


5.2 TWO- AND THREE-VARIABLE KARNAUGH MAPS

- If a function has two or more minimum sum-of-products forms, all of these forms can be determined from a map.
- Figure 5-9 shows the two minimum solutions for

$$F = \Sigma m(0, 1, 2, 5, 6, 7).$$

FIGURE 5-9: Function with Two Minimum Forms



5.3 FOUR-VARIABLE KARNAUGH MAPS

- Figure 5-10 shows the location of **minterms** on a **four-variable** map.
- Each **minterm** is located adjacent to the **four terms** with which it can combine.
- **For example**, m_5 (0101) could combine with m_1 (0001), m_4 (0100), m_7 (0111), or m_{13} (1101) because it differs in only one variable from each of the other minterms.
- The definition of adjacent squares must be extended so that not only are top and bottom rows adjacent as in the three-variable map, but the first and last columns are also adjacent

5.3 FOUR-VARIABLE KARNAUGH MAPS

- This requires numbering the columns in the sequence 00, 01, 11, 10 so that minterms 0 and 8, 1 and 9, etc., Are in adjacent squares.

FIGURE 5-10
Location of Minterms on Four-Variable Karnaugh Map

	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

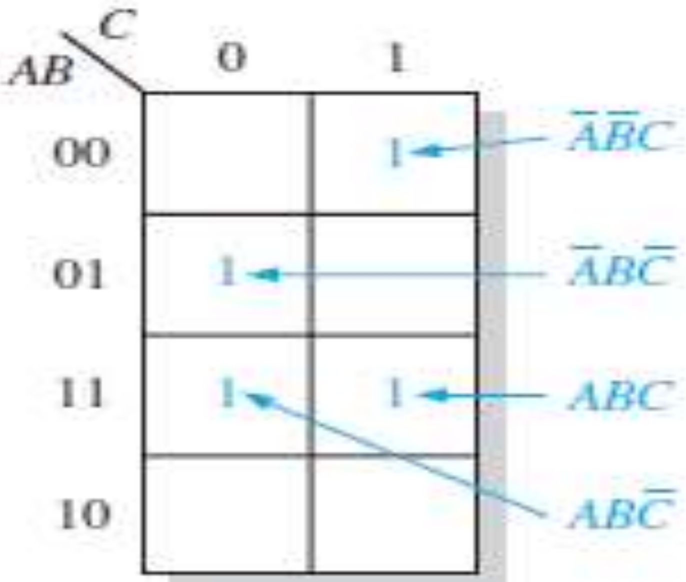
EXAMPLE_1

- Map the following standard SOP expression on a karnaugh map:

  A'B'C + A'BC' + ABC' + ABC

Evaluate the expression as shown below. Place a 1 on the 3-variable Karnaugh map in Figure 4–29 for each standard product term in the expression.

$$\begin{array}{ccccccc} \overline{A}\,\overline{B}C & + & \overline{A}B\overline{C} & + & AB\overline{C} & + & ABC \\ 001 & & 010 & & 110 & & 111 \end{array}$$



EXAMPLE_2

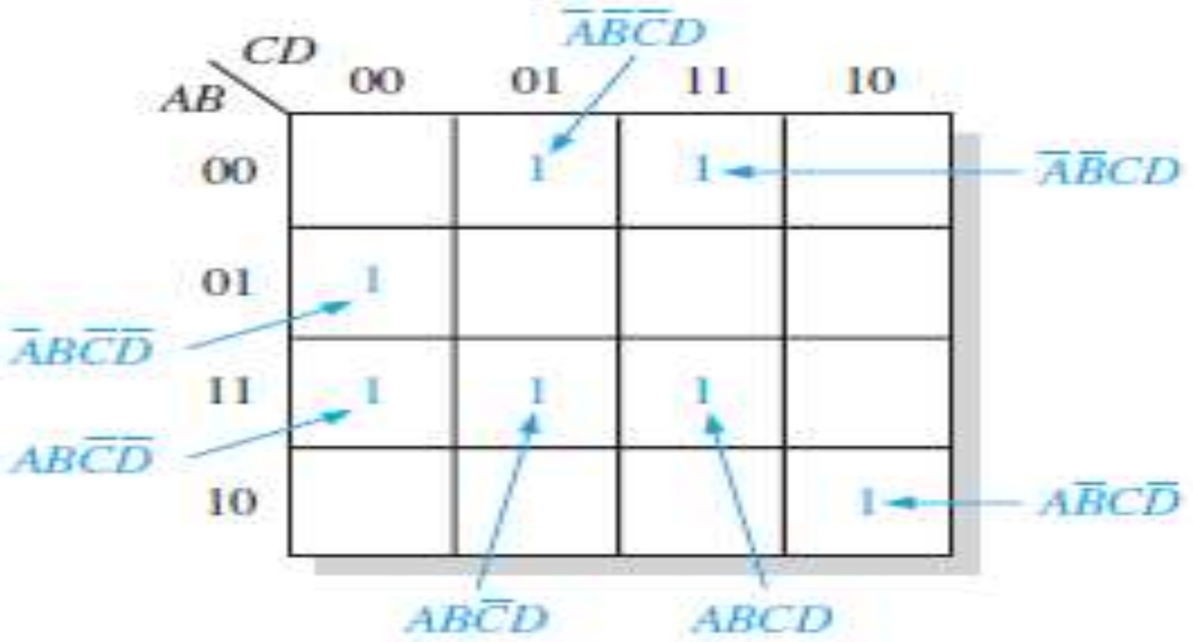
- Map the following standard SOP expression on a karnaugh map:

$$\overline{A}\,\overline{B}CD + \overline{A}B\overline{C}\,\overline{D} + AB\overline{C}D + ABCD + AB\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + A\overline{B}C\overline{D}$$

Evaluate the expression as shown below. Place a 1 on the 4-variable Karnaugh map in Figure 4–30 for each standard product term in the expression.

$$\underset{0011}{\overline{A}\,\overline{B}CD} + \underset{0100}{\overline{A}B\overline{C}\,\overline{D}} + \underset{1101}{AB\overline{C}D} + \underset{1111}{ABCD} + \underset{1100}{AB\overline{C}\,\overline{D}} + \underset{0001}{\overline{A}\,\overline{B}\,\overline{C}D} + \underset{1010}{A\overline{B}C\overline{D}}$$

$\overline{A} + A\overline{B} + AB\overline{C}.$



EXAMPLE_3

- Map the following standard SOP expression on a karnaugh map:

$$\overline{A} + A\overline{B} + AB\overline{C}.$$

The SOP expression is obviously not in standard form because each product term does not have three variables. The first term is missing two variables, the second term is missing one variable, and the third term is standard. First expand the terms numerically as follows:

$\overline{A}$	$+ A\overline{B}$	$+ AB\overline{C}$
000	100	110
001	101	
010		
011		

EXAMPLE_3 CONTINUE

Map each of the resulting binary values by placing a 1 in the appropriate cell of the 3-variable Karnaugh map in Figure 4–31.

$AB \backslash C$	0	1
00	1	1
01	1	1
11	1	
10	1	1

EXAMPLE_4

- Map the following standard SOP expression on a karnaugh map:

$$\overline{B}\,\overline{C} + A\overline{B} + AB\overline{C} + A\overline{B}C\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + A\overline{B}CD$$

The SOP expression is obviously not in standard form because each product term does not have four variables. The first and second terms are both missing two variables, the third term is missing one variable, and the rest of the terms are standard. First expand the terms by including all combinations of the missing variables numerically as follows:

$\overline{B}\,\overline{C}$	+ $A\overline{B}$	+ $AB\overline{C}$	+ $A\overline{B}C\overline{D}$	+ $\overline{A}\,\overline{B}\,\overline{C}D$	+ $A\overline{B}CD$
0000	1000	1100	1010	0001	1011
0001	1001	1101			
1000	1010				
1001	1011				

EXAMPLE_4 CONTNUE

Map each of the resulting binary values by placing a 1 in the appropriate cell of the 4-variable Karnaugh map in Figure 4–32. Notice that some of the values in the expanded expression are redundant.

$AB \backslash CD$	00	01	11	10
00	1	1		
01				
11	1	1		
10	1	1	1	1

EXAMPLE 4–28

Determine the product terms for the Karnaugh map in Figure 4–35 and write the resulting minimum SOP expression.

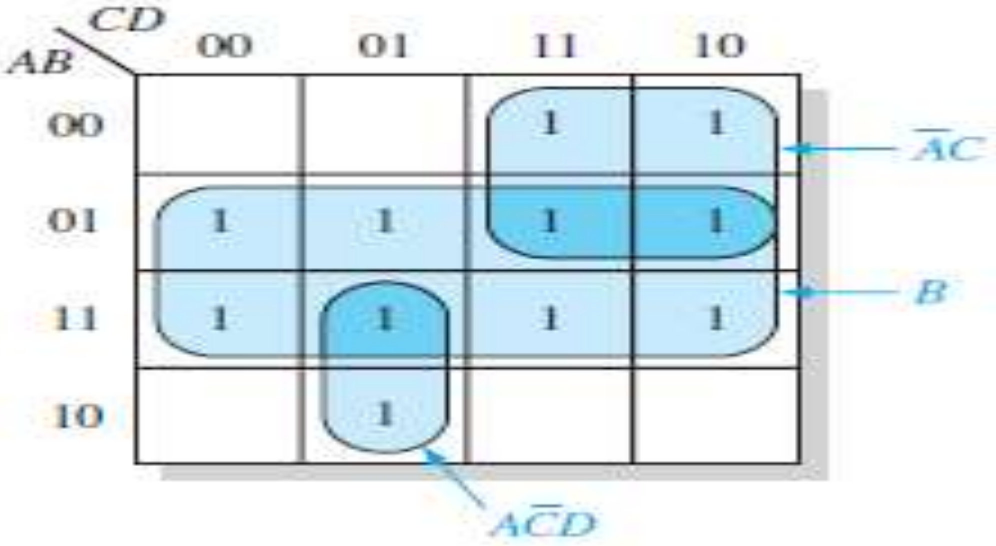


FIGURE 4–35

Solution

Eliminate variables that are in a grouping in both complemented and uncomplemented forms. In Figure 4–35, the product term for the 8-cell group is B because the cells within that group contain both A and $\overline{A}$, C and $\overline{C}$, and D and $\overline{D}$, which are eliminated. The 4-cell group contains B, $\overline{B}$, D, and $\overline{D}$, leaving the variables $\overline{A}$ and C, which form the product term $\overline{A}C$. The 2-cell group contains B and $\overline{B}$, leaving variables A, $\overline{C}$, and D which form the product term $A\overline{C}D$. Notice how overlapping is used to maximize the size of the groups. The resulting minimum SOP expression is the sum of these product terms:

$$B + \overline{A}C + A\overline{C}D$$

Use a Karnaugh map to minimize the following standard SOP expression:

$$A\overline{B}C + \overline{A}BC + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C}$$

Solution

The binary values of the expression are

$$101 + 011 + 001 + 000 + 100$$

Map the standard SOP expression and group the cells as shown in Figure 4–37.

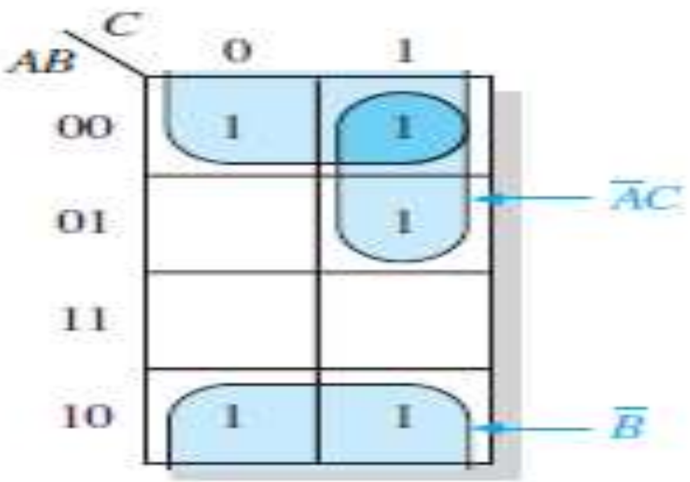


FIGURE 4–37

Notice the "wrap around" 4-cell group that includes the top row and the bottom row of 1s. The remaining 1 is absorbed in an overlapping group of two cells. The group of four 1s produces a single variable term, $\overline{B}$. This is determined by observing that within the group, $\overline{B}$ is the only variable that does not change from cell to cell. The group of two 1s produces a 2-variable term $\overline{A}C$. This is determined by observing that within the group, $\overline{A}$ and C do not change from one cell to the next. The product term for each group is shown. The resulting minimum SOP expression is