4.3 The Phase Plane for Linear Systems of Differential Equations

4.3.1 Introduction to the Phase Plane for Linear Systems of Differential Equations

In this section, we continue to analyze only linear systems of differential equations,

$$\frac{dx}{dt} = ax + by, \tag{1a}$$

$$\frac{dy}{dt} = cx + dy. \tag{1b}$$

This section is devoted to explaining the important concept of the **plane plane**. But in this section we only consider the phase plane for linear systems (1a)–(1b). The phase plane for nonlinear systems is discussed in Chapter 6.

Solutions $x(t)$, $y(t)$ of differential equations were previously each graphed as functions of time. Here instead we introduce the x, y-plane. Each value of t corresponds to a point x, y in the plane. A solution of the differential equation $x(t)$, $y(t)$ satisfying a given initial condition now traces out a curve in the x, y-plane. This parameterized curve (along with an indication of the direction the solution moves along the curve as time t increases) is called a **trajectory** or **orbit**. (The term **solution curve** is also sometimes used.) The set of trajectories (corresponding to all initial conditions) in the x, y-plane, together with an indication of the solutions direction as time increases, is the **phase plane**. Usually only a few representative solutions are drawn. (This sketch is sometimes called a phase portrait.)

Example 4.3.1 *Trajectory in the Phase Plane*

We consider the linear system

$$\frac{dx}{dt} = y, \tag{2a}$$

$$\frac{dy}{dt} = -x. \tag{2b}$$

● SOLUTION. Although we can solve this system by matrix methods, in this example, elimination is perhaps simpler. Substituting $y = \frac{dx}{dt}$ into (2b) yields the second-order differential equation with constant coefficients

$$\frac{d^2x}{dt^2} = -x. \tag{3}$$

The characteristic equation obtained by substituting $x = e^{rt}$ is $r^2 = -1$, so that $r = \pm i$ and the general solution is

$$x = c_1 \cos t + c_2 \sin t, \tag{4a}$$

$$y = -c_1 \sin t + c_2 \cos t. \tag{4b}$$

We wish to consider the solution associated with one initial condition, and for simplicity we choose $x(0) = 1$ and $y(0) = 0$. In this case $c_1 = 1$ and $c_2 = 0$, so the solution

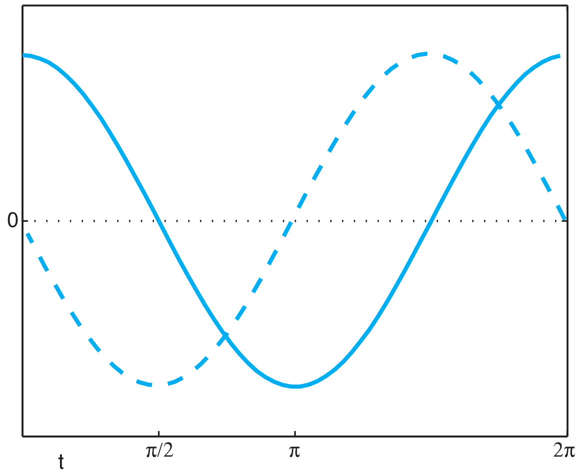

Figure 4.3.1 Solution of initial value problem $x(t) = \cos t$, $y(t) = -\sin t$.

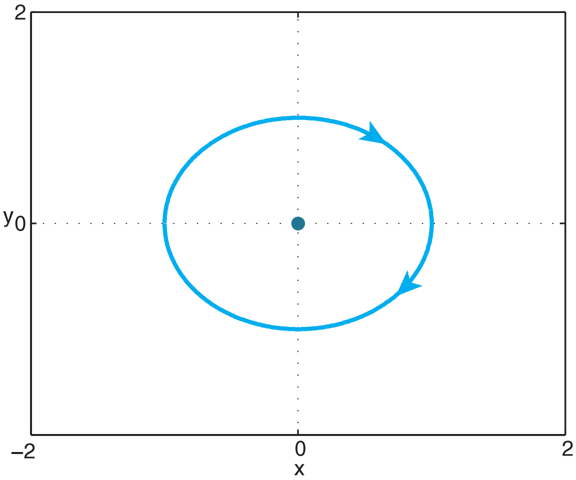

Figure 4.3.2 Trajectory in the phase plane.

of this initial value problem is

$$x = \cos t \tag{5a}$$
$$y = -\sin t. \tag{5b}$$

In figure 4.3.1we graph (5a)–(5b) in the traditional way $x = x(t)$ and $y = y(t)$, giving two elementary trigonometric functions. The phase plane (x, y) results by just plotting the set of points $(x(t), y(t))$ for different values of t. Thus the figure does not show t. The trajectory for this initial condition is the circle (graphed in figure 4.3.2) with radius 1, since

$$x^2 + y^2 = 1. \tag{6}$$

By plotting points for different t values, we see the trajectory moves clockwise. For example, $t = 0$ corresponds to $x = 1$, $y = 0$ and $t = \pi/2$ corresponds to $x = 0$, $y = -1$. The reason the solution moves clockwise in this example is that here the usual polar angle θ is $\theta = -t$, and thus the polar angle decreases in time. Arrows indicating how the solution moves clockwise, as time t increases is also included in figure 4.3.2. We will soon give further discussion on the direction of the orbits in the phase plane. ♦

PHASE PLANES ON COMPUTER SCREENS. It is quite common now for phase planes to be readily determined and vividly displayed on your computer screen using software such as Matlab, Scilab, Mathematica, or MAPLE. You will probably learn more about the subject if you have such a system available to you, or at least have access to a graphics program for phase planes for differential equations. These programs solve the system of differential equations numerically subject to given initial conditions. It is then easy for the program to take the numerical time-dependent solutions for $x(t)$ and $y(t)$ and use them directly to graph the phase plane (x, y). The time-dependent solutions of the differential equation are the parametric description of the curve shown on your screen which we call the phase plane.

NUMERICAL SOLUTIONS. Here we are not particularly interested in what numerical method the program uses. That is a separate interesting specialized mathematical subject. Those interested in numerical methods for differential equations should consult longer books on differential equations or specialized books on numerical methods for ordinary differential equations.

DIRECTION FIELD. The phase plane can be graphically constructed directly from the system of differential equations (1a)–(1b) without solving the system of differential equation. The vector $\mathbf{x} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ has the representation as a row vector $\mathbf{x} = (x, y)$ or $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$, so that in calculus it is shown that $\frac{d\mathbf{x}}{dt}$ is a tangent vector to the orbit or trajectories. As with using the slope to help sketch the solutions for first-order differential equations (see Section 1.4), a grid of points is chosen. At each (x, y) point in the grid, the right hand side of the system of differential equations (1a)–(1b) immediately determines the vector

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix},$$

which is tangent to the solution curves in the phase plane in the direction that time increases.

If software plots these vectors, the plot is called a **vector field** plot. However, often some of the vectors are small, and others large, so that it can be hard to tell what is happening to solutions in some regions. Accordingly, many programs have the ability to plot the **direction field** for systems, including the linear systems we study in this chapter. This is easy to implement on computers or even graphing calculators. It is similar to a velocity vector, and everywhere (at each point in the grid) it has magnitude and direction. Usually the program draws a vector of equal small size so as not to interfere with neighboring vectors, and we call this the **direction field.** By having all vectors the same size it is easier to visualize the solutions. The phase plane may be

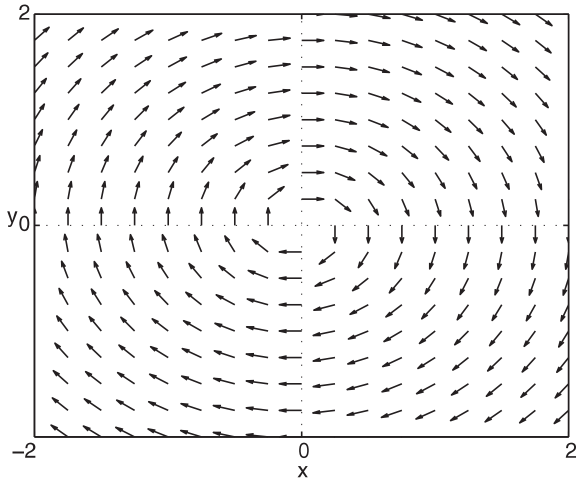

Figure 4.3.3 Direction field for (2a)–(2b).

approximated by forming curves tangent to the vectors of the direction field. We will do some examples.

EXAMPLE OF DIRECTION FIELD. Consider the previous example of a linear system (2a)–(2b). The direction field for this linear system is shown in figure 4.3.3 from some computer output. Since this is the first example for which we do a direction field, we want to explain very carefully. Let us take, for example, $x = 1$ and $y = 1$. From the differential equation, at that point the direction field should be a vector in the direction $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. Look at figure 4.3.3 at the point $x = 1$, $y = 1$, and note the vector there is in that direction. For example, for the point $x = -1$ and $y = -1$, from the differential equation the direction field should be a vector in the direction $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$. We also compare the phase plane of the exact solution (circle) with the direction field and we see that the direction field suggests motion similar to the circle. The direction field is tangential to the circular orbits in the phase plane.

EQUILIBRIUM SOLUTION. The origin $x(t) = 0$, $y(t) = 0$ is a constant or **equilibrium** solution (does not depend on t) of any linear (homogeneous) system (1a)–(1b). Solutions of the linear system whose initial conditions are at the origin stay at the origin. The phase plane representation for the equilibrium solution is just a point that does not move, the equilibrium at the origin. For solutions of the system of differential equations that are not equilibria, the solutions will move in time. Then the trajectory or orbits in the phase plane will be curves.

STABLE OR UNSTABLE EQUILIBRIUM. If all solutions of the linear system stay near the equilibrium for all initial conditions near the equilibrium, then we say the equilibrium is **stable**. If there is at least one initial condition for which the solution goes away from the equilibrium, then we say the equilibrium is unstable. In this section, we will determine conditions for which the equilibrium (the origin) for the linear system is stable or unstable.

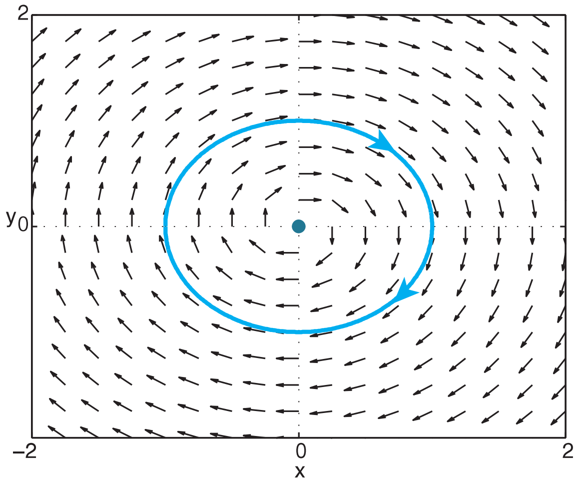

Figure 4.3.4 Trajectory in phase plane sketched from direction field.

OVERVIEW. In this section our interest is in explaining or directly constructing the phase planes using mathematical ideas. We will show how to obtain these trajectories in the phase plane using explicit solutions of the system of linear differential equations by using the matrix methods for solutions to linear systems of differential equations we have just studied in Section 4.2. Eigenvalues and eigenvectors will be very important. We will learn how to solve the phase plane for linear systems by doing a large number of examples. We begin with some very elementary examples where we do not need matrix methods. Then, we proceed to systematically investigate most cases of linear systems of differential equations including those involving real and complex eigenvalues.

In the problems of this section, $\frac{d\mathbf{x}}{dt}$ does not depend explicitly on t. Thus if two trajectories were to intersect, then at that point there would be two solutions which satisfy the same initial condition which would violate the uniqueness of solutions. Thus

Trajectories cannot intersect other trajectories or cross themselves.

This fact will be used repeatedly in what follows.

Example 4.3.2

For the following linear system, (a) identify its equilibrium; (b) sketch the direction field using software; (c) explain the direction field using the system of differential equations; and (d) sketch the phase plane using solutions of the system of differential equations;

$$\frac{dx}{dt} = -3x, \tag{7a}$$

$$\frac{dy}{dt} = -y. \tag{7b}$$

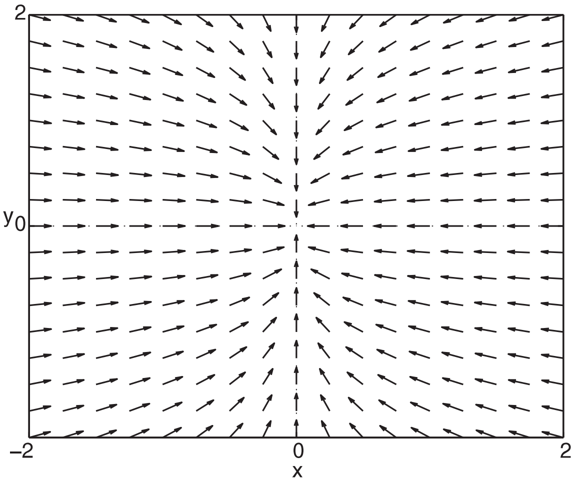

Figure 4.3.5 Direction field for (7a)–(7b).

• SOLUTION. (a) Here $x = y = 0$ solves the system of differential equations, so $(0, 0)$ is an equilibrium point. It is the only solution of $-3x = 0$, $-y = 0$, so it is the only equilibrium solution.

(b) Figure 4.3.5 shows the direction field (based on some readily available computer program) for the time-dependent system (7a)–(7b). (c) Since $\frac{dx}{dt} = -3x$, in the right half-plane (where $x > 0$) trajectories satisfy $\frac{dx}{dt} < 0$, so $x(t)$ decreases as time increases and the solution flows to the left. Similarly, in the left half-plane (where $x < 0$) trajectories flow to the right. Since $\frac{dy}{dt} = -y$, trajectories in the upper half-plane (where $y > 0$) have $\frac{dy}{dt} < 0$, so $y(t)$ decreases as time increases and the solution flows downward. In the lower half-plane ($y < 0$) the solution moves upward at time increases. From the figure we see that all solutions "flow into" the equilibrium $(0, 0)$. Such an equilibrium is called **asymptotically stable**.

(d) We also can explain the trajectories in the phase plane using the explicit solution of the system. The system (7a)–(7b) constitutes an uncoupled pair of linear differential equations whose solutions are

$$x(t) = c_1 e^{-3t}, \quad y(t) = c_2 e^{-t}. \tag{8}$$

If $c_2 = 0$, the solution in the phase plane is simple, namely, the line $y = 0$. Since $x(t) = c_1 e^{-t}$, the trajectories along $y = 0$ approach the origin. More precisely, the trajectories correspond to two rays approaching the origin (one with positive x and one with negative x) and the equilibrium. Similarly, the solution corresponding to $c_1 = 0$ corresponds to two rays approaching the origin (in opposite directions) along $x = 0$. These are the four straight rays sketched in figure 4.3.6. If both $c_1 \neq 0$ and $c_2 \neq 0$, then the trajectory is more complicated. Certainly, all solutions approach the origin as $t \to +\infty$. As $t \to +\infty$, $e^{-3t} \to 0$ much faster than e^{-t}, so that these other trajectories must approach and be tangent to the line $x = 0$ as $t \to +\infty$, as shown in figure 4.3.6 This kind of equilibrium is an example of a **stable node**. We will discuss other examples of this later in this section. ♦

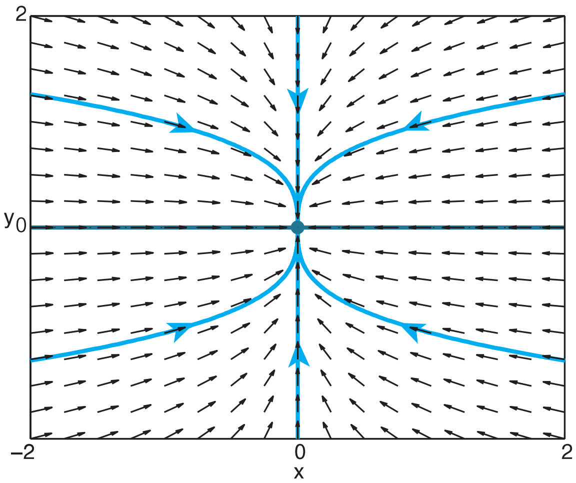

Figure 4.3.6 Phase plane for (7a)–(7b) sketched with direction field.

Example 4.3.3

Sketch the direction field and the phase plane for the system

$$\frac{dx}{dt} = 3x, \tag{9a}$$

$$\frac{dy}{dt} = y, \tag{9b}$$

and describe the behavior of the solutions near the equilibrium (0, 0).

● SOLUTION. It turns out that the direction field and phase plane for this example is almost identical to the previous one. The direction field and phase plane are the same but the directions of the arrows in time are reversed. To show that, we let $t = -\tau$. Since, for example, by the chain rule $\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = -\frac{dx}{dt}$. The system of differential equations becomes

$$\frac{dx}{d\tau} = -3x, \tag{10a}$$

$$\frac{dy}{d\tau} = -y. \tag{10b}$$

The trajectories are the same, but their directions are reversed in time. Now all solutions flow away from the equilibrium (0, 0); the equilibrium is **unstable**. This kind of equilibrium is called an **unstable node**. All solutions (except the equilibrium itself) go away from the equilibrium (origin) as time increases, and we note that the solutions go to infinity as $t \to +\infty$. It is most important to note that all solutions approach the origin backward in time (as $t \to -\infty$). A more subtle result is to note that backward in time the solution approaches the origin tangent to $x = 0$. ⧫

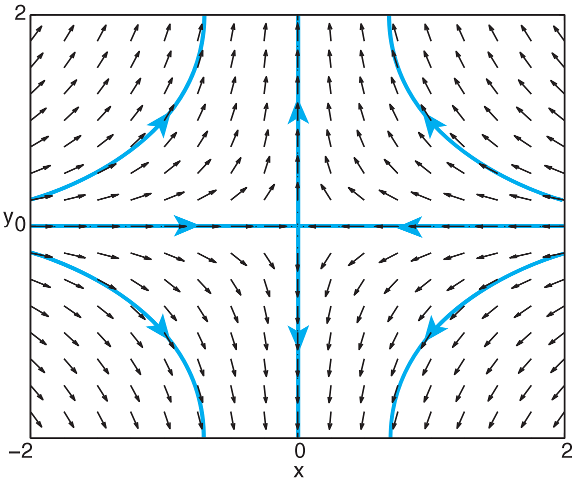

Figure 4.3.7 Phase plane for (11a)–(11b) sketched with direction field.

Example 4.3.4

Sketch the direction field using software and sketch the phase plane for the linear system of differential equations using solutions of the system of differential equations

$$\frac{dx}{dt} = -x, \tag{11a}$$

$$\frac{dy}{dt} = 2y. \tag{11b}$$

● SOLUTION. The equilibrium is again the origin $x = y = 0$ or $(0, 0)$ The direction field is sketched in figure 4.3.7. The solution to the system is $x(t) = c_1e^{-t}$ and $y(t) = c_2e^{2t}$. If $c_2 = 0$, the solution in the phase plane is again the line $y = 0$. Since $x(t) = c_1e^{-t}$, the trajectories along $y = 0$ approach the origin. More precisely, the trajectories correspond to two rays approaching the origin, exactly the same as Example 4.3.2. The solution corresponding to $c_1 = 0$ corresponds to the vertical line $x = 0$, but arrows are introduced along $x = 0$ away (outgoing) from the origin, since y is exponentially increasing and going toward ± infinity. This solution $x = 0$ with $y(t) = c_2e^{2t}$ approaches the origin backward in time (as $t \to -\infty$). There are four straight line rays sketched in figure 4.3.7, two going toward the origin but two away from the origin. If both $c_1 \neq 0$ and $c_2 \neq 0$, then the trajectory is more complicated. These solutions approach infinity as $t \to +\infty$ in the direction $x = 0$. These solutions also approach infinity backward in time as $t \to -\infty$, but along $y = 0$, as shown in figure 4.3.7. This kind of equilibrium is an example of a **saddle point**. We will discuss other examples of this later in this section. ♦

An equilibrium is defined to be stable if all initial conditions near the equilibrium stay near the equilibrium. (A more technical definition is needed for nonlinear systems.) In this example, initial conditions along $y = 0$ approach the equilibrium in a stable-like manner. However, the definition requires this to occur for all initial conditions. We can see from the figure that most initial conditions go away from the equilibrium, so we say that **all saddle points are unstable equilibrium**.

The last three examples were intended as motivation for a more general discussion of phase plane analysis for linear systems of differential equations

$$\frac{dx}{dt} = ax + by, \tag{12a}$$

$$\frac{dy}{dt} = cx + dy. \tag{12b}$$

4.3.2 Phase Plane for Linear Systems of Differential Equations

For the rest of this section, we continue to study linear (homogeneous) systems of differential equations

$$\frac{dx}{dt} = ax + by, \tag{13a}$$

$$\frac{dy}{dt} = cx + dy. \tag{13b}$$

We note that $x = y = 0$ corresponding to the origin $(0, 0)$ is always an equilibrium point.

An understanding of the phase plane of linear systems comes from their explicit solution. In Section 4.2 we showed how to use eigenvalues and eigenvectors to solve linear system. We substitute

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{14}$$

and obtain

$$(a - \lambda)u + bv = 0, \tag{15a}$$

$$cu + (d - \lambda)v = 0, \tag{15b}$$

or what we will use from now on,

$$\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{16}$$

The pair of equations has a nonzero solution for u, v if and only if the determinant of the coefficient matrix is zero, that is,

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc = 0. \tag{17}$$

This is called the **characteristic equation** for the linear system of differential equations (13a)–(13b). This is also the condition for λ to be an **eigenvalue** of the coefficient matrix $\mathbf{A} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The eigenvector $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ corresponding to the eigenvalue λ satisfies (15a)–(15b) or (16).

There are a number of questions we wish to answer:

1. Is the equilibrium $x = y = 0$ stable or unstable?
2. What happens to the solution as time increases, and in particular what happens for long time ($t \to +\infty$)?
3. What do trajectories of the solutions look like in the phase plane?

It turns out that the behavior of the solutions of linear systems of differential equations is linked to the nature of the eigenvalues λ_1 and λ_2 of the coefficient matrix. Different behavior occurs if the eigenvalues are both positive, both negative, of opposite signs, or are complex (with positive, negative, or zero real part). We consider most cases in detail by first considering a special example and then indicating what happens in general. For your convenience, at the end of this section we summarize these results in Theorem 1.

4.3.3 Real Eigenvalues

Case 1: λ_1, λ_2 ***Real, Distinct, and Positive (Unstable Node)***

Example 4.3.5

We first do an example similar to Example 4.3.1, which has both eigenvalues real, distinct, and positive:

$$\frac{dx}{dt} = x, \tag{18a}$$

$$\frac{dy}{dt} = 4y. \tag{18b}$$

● SOLUTION. The eigenvalues (roots of the characteristic equation) are 1 and 4, and the solution to the system is

$$x(t) = c_1 e^t,\ y(t) = c_2 e^{4t}. \tag{19}$$

This can be rewritten as a vector, which helps in understanding the trajectories in the phase plane:

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_2 e^{4t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}. \tag{20}$$

For example, (see figure 4.3.8), if $c_2 = 0$, the solution in the phase plane is $c_1 \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] e^t$. This solution is in the direction $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ which corresponds to the x-axis ($y = 0$), corresponding to two rays (depending on the sign of c_1) which go to infinity as time increases (and approach the origin as $t \to -\infty$). The other straight line trajectory going to infinity as time increases corresponds to the solution $c_2 \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] e^{4t}$ and is in the vertical direction. The other solutions shown in figure 4.3.8 go away from the origin and go to infinity as time increases. We say the equilibrium $(0, 0)$ is an **unstable node**. These trajectories also go to the origin as $t \to -\infty$ in the direction tangent to $y = 0$. ♦

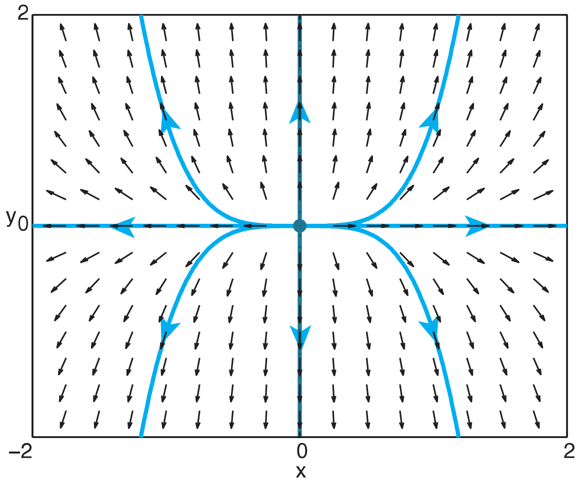

Figure 4.3.8 Phase plane for (18a)–(18b), an unstable node.

WE NOW DISCUSS CASE I IN GENERAL WITH REAL DISTINCT POSITIVE EIGENVALUES $\lambda_1 > 0$ AND $\lambda_2 > 0$ (UNSTABLE NODE). Suppose $\mathbf{v}_1$ is an eigenvector corresponding to the eigenvalue λ_1, and $\mathbf{v}_2$ is an eigenvector corresponding to the eigenvalue λ_2, as shown in Section 4.2. That means there are two elementary solutions of the system, $\mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{v}_2 e^{\lambda_2 t}$. From this we obtain the **general solution**,

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}. \tag{21}$$

If $c_2 = 0$, the solution is $\mathbf{x}(t) = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = c_1 \mathbf{v}_1 e^{\lambda_1 t}$, with $\lambda_1 > 0$. In the phase plane this solution is in the direction of the eigenvector $\mathbf{v}_1$, since the solution is time-dependent multiples of the eigenvector. In the phase plane, the trajectory of this solution is a straight line in the direction of the eigenvector $\mathbf{v}_1$, going away from the origin (since $\lambda_1 > 0$) toward infinity. These correspond to the two outward going rays in the direction of the eigenvector $\mathbf{v}_1$ shown in figure 4.3.9.

Similarly, if $c_1 = 0$, the solution is $\mathbf{x}(t) = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = c_2 \mathbf{v}_2 e^{\lambda_2 t}$, with $\lambda_2 > 0$. In the phase plane, this solution also corresponds to two outward going straight lines (since $\lambda_2 > 0$) trajectories in the direction of the other eigenvector $\mathbf{v}_2$, as seen in figure 4.3.9. The other trajectories in the phase plane move away in time from the origin and go to infinity. Backward in time, solutions approach the origin. **In general, when the roots are distinct, real, and positive, the origin is an unstable node whose phase plane resembles figure 4.3.9.** It can be shown (subtle) that backward in time the solution approaches the eigenvector direction of the smallest positive eigenvalue (since as $t \to -\infty$ the exponential $e^{\lambda t}$ corresponding to the largest eigenvalue goes to zero fastest). Computer-generated phase planes for linear systems are often incomplete, without the straight line trajectories corresponding to the eigenvector directions.

Case 2: λ_1, λ_2 Real, Distinct, and Negative (Stable Node)

We again start with an example.

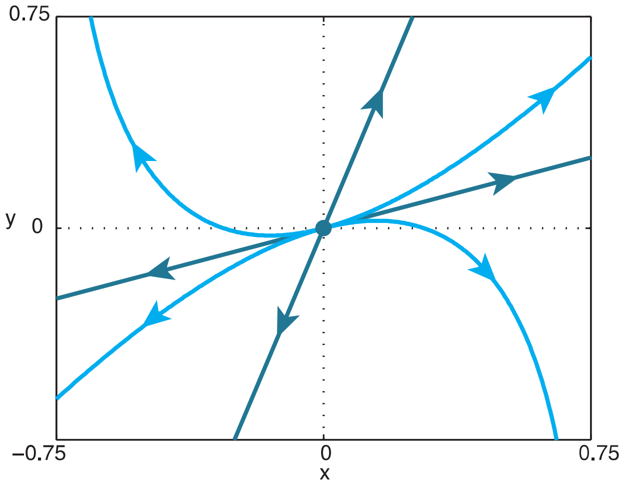

Figure 4.3.9 Phase plane for unstable node, assuming $\lambda_2 > \lambda_1 > 0$.

Example 4.3.6

We do an example similar to Example 4.2.1 with both eigenvalues real, distinct, and negative:

$$\frac{dx}{dt} = -x, \tag{22a}$$

$$\frac{dy}{dt} = -4y. \tag{22b}$$

● SOLUTION. The eigenvalues (roots of the characteristic equation) are −4 and −1, and the general solution to the system is

$$x(t) = c_1 e^{-t}, \qquad y(t) = c_2 e^{-4t}. \tag{23}$$

We can rewrite this as a vector:

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-4t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-4t}. \tag{24}$$

A phase plane diagram for this system is given in figure 4.3.10. If $c_2 = 0$, the solution in the phase plane is $c_1 \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right] e^{-t}$. This solution is two straight line rays going toward the origin along the x-axis ($y = 0$). If $c_1 = 0$, the solution in the phase plane is two vertical rays moving toward the origin. These four rays are marked in blue in figure 4.3.10. In this example, the trajectories are moving toward the origin. For this reason, the origin is called an **asymptotically stable** equilibrium. Most trajectories approach the origin along curves that are tangent to the x-axis ($y = 0$). The equilibrium at the origin is a **stable node**. ♦

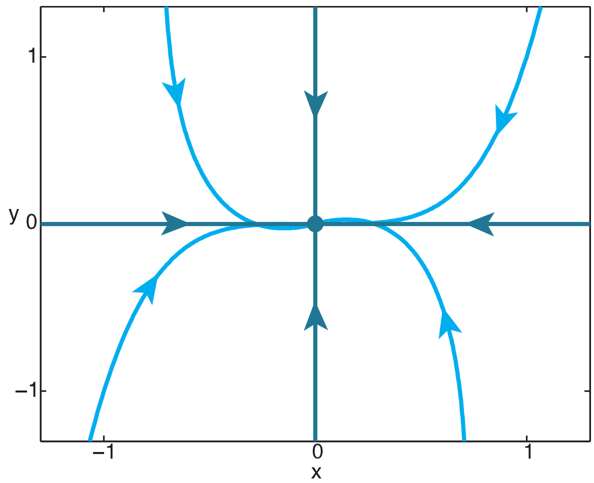

Figure 4.3.10 Phase plane for (22a)–(22b), a stable node.

Example 4.3.7 *Case 2. λ_1, λ_2 Real, Distinct, and Negative (Stable Node)*

Classify the equilibrium at the origin and sketch a phase plane for the linear system

$$\frac{dx}{dt} = -2x - 1y, \tag{25a}$$

$$\frac{dy}{dt} = -x - 2y. \tag{25b}$$

● SOLUTION. We substitute

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{26}$$

and obtain

$$\begin{bmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{27}$$

The eigenvalues satisfy the determinant condition

$$\lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1) = 0. \tag{28}$$

The origin is a stable node since the eigenvalues (roots) are $-3, -1$. To sketch the trajectories in the phase plane, we determine the solution using eigenvalues and eigenvectors. The eigenvector corresponding to $\lambda = -3$ satisfies $u - v = 0$. We choose $u = 1$, so that $v = 1$, and the eigenvector corresponding to $\lambda = -3$ is $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. In this way, we obtain the elementary solution $\mathbf{x}(t) = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = c_1 \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] e^{-3t}$. The trajectories in the phase plane are two straight line rays $(y = x)$ approaching the origin in the direction $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, as shown in figure 4.3.11. The eigenvector corresponding to $\lambda = -1$ satisfies $-u - v = 0$. We choose $u = 1$ and $v = -1$, so that the eigenvector

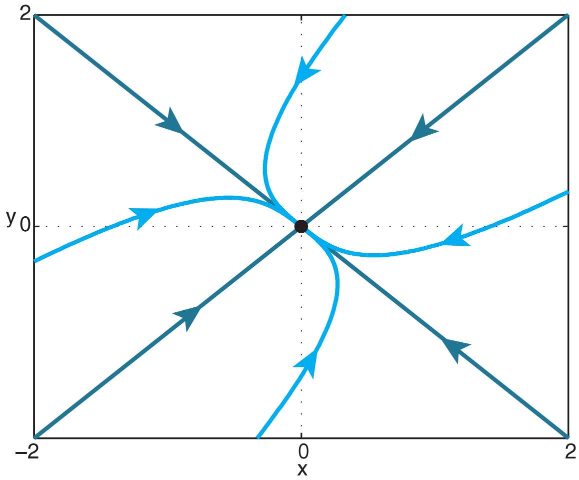

Figure 4.3.11 Phase plane for (25a)-(25b), a stable node.

corresponding to $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The elementary solution corresponding to the eigenvalue $\lambda = -1$ is $\begin{bmatrix} x \\ y \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$, which is the direction $y = -x$ of the two straight line rays approaching the origin in figure 4.3.11. From this we obtain the general solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}. \tag{29}$$

The solutions approach the origin as time increases. More specifically, the non-straight line solutions approach the origin tangent to $y = -x$ (same as the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$) since e^{-3t} is much smaller than e^{-t} as $t \to +\infty$. ♦

WHEN THE ROOTS (EIGENVALUES) ARE DISTINCT, REAL, AND NEGATIVE, CASE 2 THE ORIGIN IS A STABLE NODE. The phase plane near the origin resembles that in figure 4.3.11). Cases 1 and 2 are quite similar. They differ mainly in that when the roots (eigenvalues) are both positive, the trajectories move away from the origin (the origin is unstable), while when the roots are both negative, the trajectories approach the origin (the origin is stable). The general solution is again given by (21).

Case 3: λ_1, λ_2 *Real, Opposite Signs (Saddle Point)*

Example 4.3.8

A simple example of Case 3 with eigenvalues with real opposite signs is the system

$$\frac{dx}{dt} = 3x, \tag{30a}$$

$$\frac{dy}{dt} = -y. \tag{30b}$$

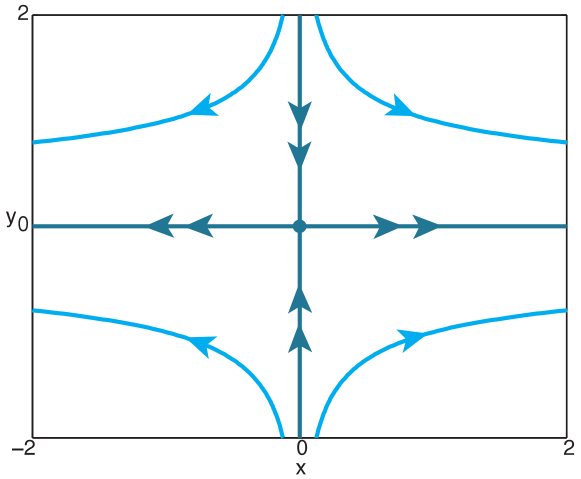

Figure 4.3.12 Phase plane for (30a)–(30b), a saddle point.

● SOLUTION. The eigenvalues (roots of characteristic equation) are 3 and -1, and the general solution to the system is

$$x(t) = c_1 e^{3t}, \quad y(t) = c_2 e^{-t}. \tag{31}$$

We can rewrite this as a vector:

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}. \tag{32}$$

The trajectories in the phase plane for this system are given in figure 4.3.12. If $c_2 = 0$, the solution in the phase plane is $c_1 \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right] e^t$. This solution is two straight line rays going away from the origin along the x-axis $y = 0$. If $c_1 = 0$, the solution in the phase plane is two vertical rays moving toward the origin inward along the y-axis ($x = 0$). These four rays are marked in figure 4.3.12. If $c_1 \neq 0$ and $c_2 \neq 0$, then as time increases the solution must go away from the origin. Specifically, as $t \to +\infty$, $y(t) \to 0$ but $|x(t)| \to \infty$. As time goes backward, these solutions also go away from the origin and go to infinity. Specifically, as $t \to -\infty$, $x(t) \to 0$ but $|y(t)| \to \infty$. Trajectories come in along the positive and negative y-axis and go out along the positive and negative x-axis. The origin is unstable, since there are trajectories that start near the origin but eventually move away. (Note, there are two trajectories on the y-axis which do approach the origin, but they are the only ones.) The phase plane attained with the help of the direction field is shown in figure 4.3.12. The origin is said to be a **saddle point**. It is called a saddle point, because a property of a saddle of a horse is that in one direction the saddle goes away from the seat and in the other direction it goes toward the seat. The phase plane also looks like the topographic map for a pass through a mountain. ♦

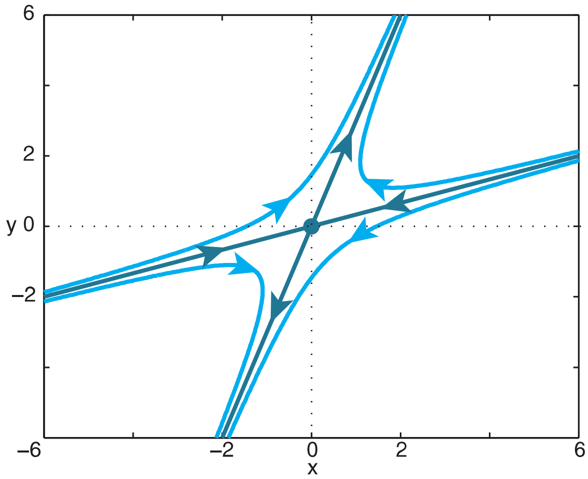

Figure 4.3.13 Phase plane for a saddle point.

CASE 3: WHEN EIGENVALUES (ROOTS) ARE REAL AND OF OPPOSITE SIGNS, THE ORIGIN IS AN UNSTABLE SADDLE POINT. The general solution is again given by (21). Saddle points are always unstable. In general, the trajectories come in alongside one eigenvector (corresponding to the negative eigenvalue) and go out alongside the other eigenvector (corresponding to the positive eigenvalue). This is illustrated in figures 4.3.13. The next example helps our understanding of the phase plane for a saddle point.

Example 4.3.9 *Case 3. λ_1, λ_2 Real, Opposite Signs (phase plane of a saddle point)*

Classify the equilibrium at the origin and sketch a phase diagram for the linear system

$$\frac{dx}{dt} = -7x + 6y, \tag{33a}$$

$$\frac{dy}{dt} = 6x + 2y. \tag{33b}$$

- SOLUTION. For systems we substitute:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix} \tag{34}$$

and obtain

$$\begin{bmatrix} -7-\lambda & 6 \\ 6 & 2-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{35}$$

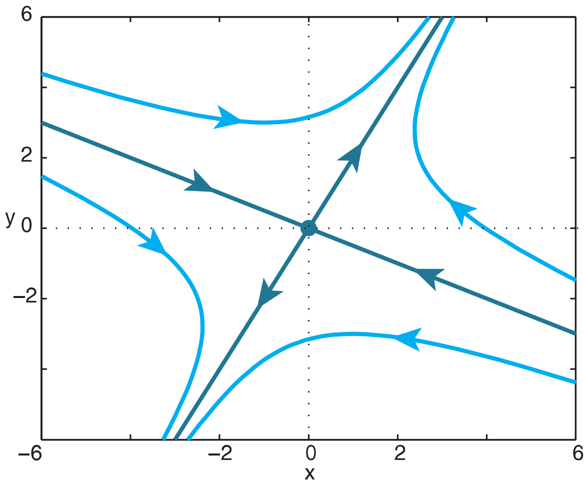

Figure 4.3.14 Phase plane for (33a)–(33b), which is a saddle point

The eigenvalues satisfy the determinant condition

$$\lambda^2 + 5\lambda - 50 = (\lambda + 10)(\lambda - 5) = 0. \tag{36}$$

The origin is an (unstable) saddle point, since the eigenvalues (roots) have opposite signs $5, -10$. To sketch the trajectories in the phase plane, we determine the solution using eigenvalues and eigenvectors. The eigenvector corresponding to $\lambda = 5$ satisfies $-12u + 6v = 0$. We choose $u = 1$, so that $v = 2$, and the eigenvector corresponding to $\lambda = 5$ is $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$. In this way, we obtain the elementary solution $\mathbf{x}(t) = \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = c_1\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]e^{5t}$. The trajectories in the phase plane are two straight line rays ($y = 2x$) going away from the origin toward infinity, in the direction $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, as shown in figure 4.3.14. The eigenvector corresponding to $\lambda = -10$ satisfies satisfies $3u + 6v = 0$. We choose $v = 1$ and $u = -2$, so that the eigenvector corresponding to $\lambda = -10$ is $\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$. The elementary solution corresponding to the eigenvalue $\lambda = -10$ is $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = c_2\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]e^{-10t}$, which is the direction $y = -x/2$ of the two straight line rays approaching the origin in Figure 4.3.14. Saddle points for linear systems are characterized by two opposite rays approaching the equilibrium but two opposite side rays going away from the equilibrium. From these solutions, we obtain the general solution:

$$\begin{bmatrix}x\\y\end{bmatrix} = c_1\begin{bmatrix}1\\2\end{bmatrix}e^{5t} + c_2\begin{bmatrix}-2\\1\end{bmatrix}e^{-10t}. \tag{37}$$

The non-straight line trajectories have the following properties. These trajectories all go away from the origin as time increases. Specifically, the non-straight line solutions approach infinity along the direction of the ray (eigenvector) associated with the positive eigenvalue, as shown in figure 4.3.14. Backward in time, these solutions approach infinity along the direction of the ray (eigenvector) associated with the negative eigenvalue. ♦

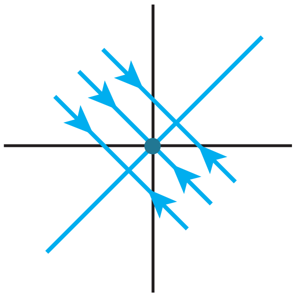

Figure 4.3.15 Case 5. Phase plane with one zero eigenvalue.

Case 4: Equal Eigenvalues (repeated roots) $\lambda_1 = \lambda_2$

We don't think this case is as important as the others, so we omit phase portraits of the two possibilities (one independent eigenvector and two independent eigenvectors).

Case 5: One Eigenvalue Zero ($\lambda_1 = 0$)

There are two cases, depending on whether $\lambda_2 < 0$ or $\lambda_2 > 0$. We just consider one example. Suppose $\lambda_1 = 0$ with eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda_2 = -10$ with eigenvector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Then the general solution would be

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-10t}. \tag{38}$$

If $c_1 = 0$, solutions along the eigenvector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ move to the origin as time increases. If $c_2 = 0$, solutions along the eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ do not move in time. Other solutions approach the line $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ parallel to the direction $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ as shown in figure 4.3.15. If the negative eigenvalue were positive, trajectories would move in the opposite direction away from the line $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

4.3.4 Complex Eigenvalues

Cases 6 and 7: Complex Eigenvalues (nonzero real part) $\lambda = \alpha \pm i\beta$*: Spirals*

We now consider the case of complex roots $\lambda = \alpha \pm i\beta$, where $\alpha \neq 0$ and $\beta \neq 0$. (Pure imaginary roots, $\lambda = \pm i\beta$, are discussed in Case 8.) For complex roots, because of Euler's formula, the solutions involve $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$. It is clear that when $\alpha > 0$, the trajectories will travel away from the origin. Hence, the origin is unstable. But when $\alpha < 0$, the trajectories approach the origin and the origin is asymptotically stable. The result in the phase plane is a **spiral** centered at the origin, and the proof in some important special cases is given below. **Case 6:** If $\alpha > 0$, then the equilibrium is an **unstable spiral**. **Case 7:** If $\alpha < 0$, then the equilibrium is a **stable spiral**. The solutions may spiral either clockwise or counterclockwise but that is not usually an important consideration. Figure 4.3.16 shows all four possibilities. The figure shows only one spiral in each case corresponding to one initial condition. To account for all initial conditions, one should visualize an infinite number of spirals.

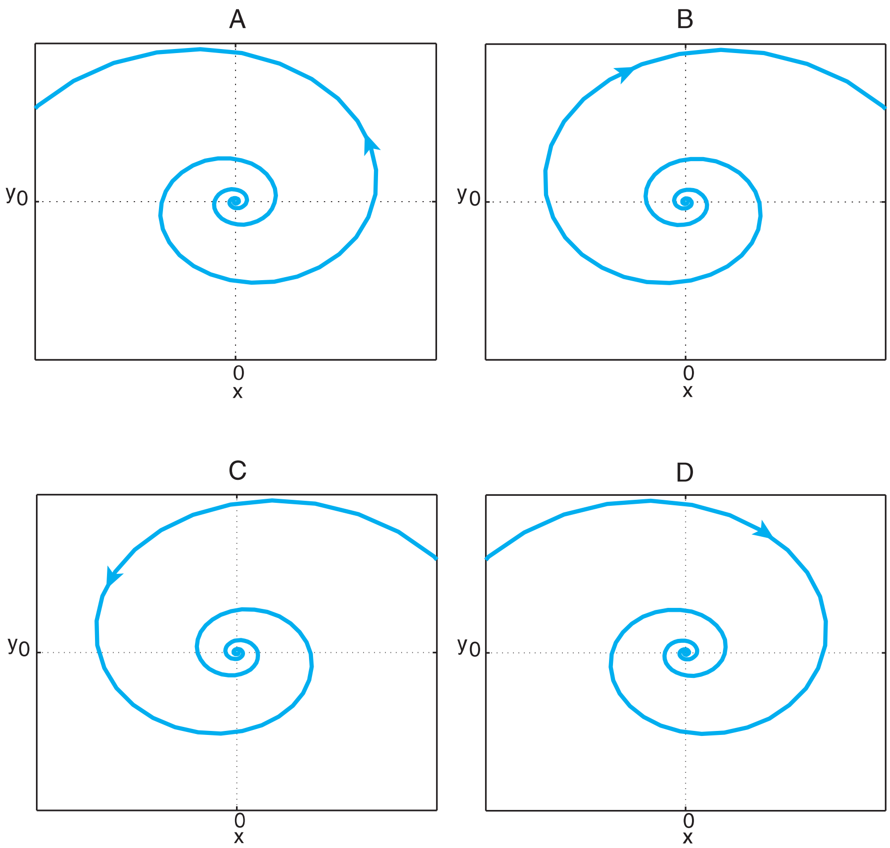

Figure 4.3.16 (a) Unstable spiral (counterclockwise), (b) unstable spiral (clockwise), (c) stable spiral (counterclockwise), (d) stable spiral (clockwise).

Example 4.3.10 *Phase Plane of a Spiral*

Determine the phase plane for

$$\frac{dx}{dt} = 2x + y, \tag{39a}$$

$$\frac{dy}{dt} = -x + 2y. \tag{39b}$$

● SOLUTION. The eigenvalues satisfy

$$\det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 5 = 0, \tag{40}$$

so that the eigenvalues (roots) are complex, $\lambda = 2 \pm i$. Since $\alpha = 2 > 0$, we have an unstable spiral. To determine whether it is clockwise or counterclockwise, we just take one simple nonzero point on the x or y axis. For example, say $x = 1$, $y = 0$, in which case the tangent vector is $[\frac{dx}{dt}, \frac{dx}{dt}] = [2, -1]$, which points down and to the right from $(1, 0)$. Thus, the unstable spiral is clockwise as in figure 4.3.16b. ♦

SOME VERY IMPORTANT EXAMPLES OF COMPLEX EIGENVALUES: CASES 6 AND 7. These examples simplify using polar coordinates. But this does not always work. To better describe the behavior of trajectories, we analyze the specific system

$$\frac{dx}{dt} = \alpha x - \beta y, \tag{41a}$$

$$\frac{dy}{dt} = \beta x + \alpha y. \tag{41b}$$

The eigenvalues satisfy

$$\det \begin{bmatrix} \alpha - \lambda & -\beta \\ \beta & \alpha - \lambda \end{bmatrix} = (\alpha - \lambda)^2 + (\beta)^2 = 0, \tag{42}$$

so that the eigenvalues (roots) are complex, $\lambda = \alpha \pm i\beta$.

For this example (but not necessarily other examples), the differential equation simplifies using the polar coordinates, $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. We obtain

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = x(\alpha x - \beta y) + y(\beta x + \alpha y) = \alpha(x^2 + y^2) = \alpha r^2, \tag{43}$$

dividing by 2 and using the differential equation. We have

$$\frac{dr}{dt} = \alpha r, \text{ with the general solution } r(t) = r(0)e^{\alpha t}. \tag{44}$$

We next obtain the differential equation for the polar angle,

$$\sec^2\theta \frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2} = \frac{x(\beta x + \alpha y) - y(\alpha x - \beta y)}{x^2} = \frac{\beta(x^2 + y^2)}{x^2}. \tag{45}$$

Since $\sec^2\theta = \frac{1}{\cos^2\theta} = \frac{r^2}{x^2}$, we obtain

$$\frac{d\theta}{dt} = \beta, \text{ with the general solution obtained by integration, } \theta(t) = \beta t + \theta(0). \tag{46}$$

Again, we see that if $\alpha > 0$, the solution moves away from the origin, since $r \to +\infty$ as $t + \infty$. But, because θ is a linear function of t, the trajectories spiral around the origin. Similarly, when $\alpha < 0$, the solutions spiral in toward the origin. We therefore refer to the origin as a **spiral point**. It is a **stable spiral** if $\alpha < 0$, and an **unstable spiral** if $\alpha > 0$. We can determine the direction of rotation of the spiral from (46). If $\beta > 0$, then θ (polar angle) increases in time, and the spiral is counterclockwise (see figure 4.3.16a or c). If $\beta < 0$, then θ decreases in time, and the spiral is clockwise (see figure 4.3.16b or d). The direction of the spiral can also be determined in a simpler way directly from the original system. For example, setting $y = 0$ in (41b), we find that $\frac{dy}{dt} = \beta x$. Now, if $\beta > 0$, then as the trajectory crosses the positive x-axis, $y = 0$, we have $\frac{dy}{dt} > 0$, and consequently, the trajectory is spiraling counterclockwise. Similarly, $\beta < 0$ the trajectory spirals clockwise. These spirals are **logarithmic spirals** because $\ln r(t) = \ln r(0) + \alpha t = \ln r(0) + \frac{\alpha}{\beta}(\theta(t) - \theta(0))$.

In the general case, the spirals may be distorted, but the stability criteria persists; when the roots $\lambda = \alpha \pm i\beta$ are complex numbers, the origin is a spiral point that is unstable when $\alpha > 0$ (see figure 4.3.16 a or b) and asymptotically stable when $\alpha < 0$ (see figure 4.3.16 c or d).

Example 4.3.11 *Case 6. Complex Eigenvalues (roots) $\lambda = \alpha \pm i\beta$: Unstable Spirals.*

Classify the equilibrium at the origin and sketch the phase plane for the system

$$\frac{dx}{dt} = 2x + y, \tag{47a}$$

$$\frac{dy}{dt} = -x + 2y. \tag{47b}$$

● SOLUTION. The eigenvalues (characteristic equation) for this system is

$$\det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 + 1 = 0,$$

which has complex eigenvalues (roots), $\lambda = 2 \pm i$. Thus, the origin is an unstable spiral, since $\alpha = 2 > 0$. Setting $y = 0$, $x = 1$ in (47b), we get the tangent vector

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

which points down and to the right from $(1, 0)$. Thus the unstable spiral is clockwise, and we have precisely the case of figure 4.3.16 b. ♦

Case 8: Purely Imaginary Eigenvalues (roots) $\lambda = \pm i\beta$: Centers

Example 4.3.12 *Undamped Spring-Mass System*

Sketch the trajectories in the phase plane for the first-order system corresponding to the unforced undamped spring mass system in Section 2.5:

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{48}$$

● SOLUTION. We will solve this oscillation of a spring mass system in a number of different ways.

SOLUTION USING SECOND-ORDER DIFFERENTIAL EQUATION METHODS. We can determine the trajectories in the phase plane by solving the second-order linear differential equation with constant coefficients associated with the spring-mass system (48). The amplitude and phase form of the general solution are particularly helpful here:

$$x(t) = A \sin(\omega t + \phi), \tag{49}$$

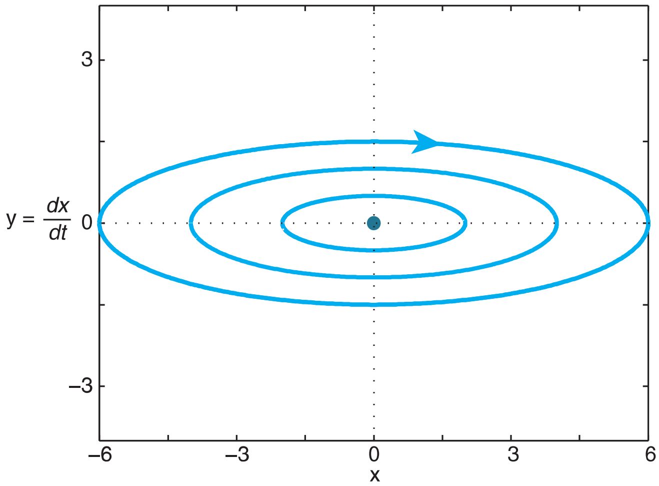

Figure 4.3.17 Phase plane for spring-mass system which is a center (ellipse).

where the natural frequency satisfies $\omega = \sqrt{\frac{k}{m}}$. In this case, by taking the derivative, the velocity $y = \frac{dx}{dt}$ satisfies

$$y(t) = \frac{dx}{dt} = A\omega \cos(\omega t + \phi). \tag{50}$$

Time can be eliminated from (49) and (50), giving directly the equation for the trajectories in the phase plane:

$$x^2 + \frac{y^2}{\omega^2} = A^2 \sin^2 + A^2 \cos^2 = A^2. \tag{51}$$

The phase plane consists of a family of **ellipses** shown in figure 4.3.17. Trajectories move clockwise since, for example, at $x = 0$, $y = 1$ from (50) we see that $\frac{dx}{dt} = 1$ so that x is increasing in time there. The solutions orbit periodically (cyclically) through the same points, with the same velocities. We call the equilibrium (0, 0) a **center**.

SOLUTION USING DIRECTION FIELD. By introducing the velocity $y = \frac{dx}{dt}$, this equation (48) can be converted to a first-order system:

$$\frac{dx}{dt} = y, \tag{52a}$$

$$\frac{dy}{dt} = -\frac{k}{m}x. \tag{52b}$$

The equilibrium is the origin $x = y = 0$. The direction field for $\frac{k}{m} = 1$ corresponds to our earlier problem, (2a), (2b). With $\omega = \sqrt{\frac{k}{m}} = 1$ the ellipses become circles as shown in figures 4.3.3 and 4.3.4. In that case the trajectories appear to be either circles or spirals which encircle the origin clockwise. They can't spiral because the exact time-dependent solutions are periodic. They are ellipses here because of (51).

SOLUTION USING SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS. For the spring-mass system, (52a)–(52b). For systems, we substitute:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{53}$$

and obtain

$$\begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{54}$$

The eigenvalues satisfy the determinant condition

$$\lambda^2 + \frac{k}{m} = 0. \tag{55}$$

The eigenvalues (roots) are purely imaginary $\lambda = \pm i\sqrt{\frac{k}{m}}$, and by Euler's formula the solution must involve $\cos\sqrt{\frac{k}{m}}t$ and $\sin\sqrt{\frac{k}{m}}t$, which we already knew. ♦

Example 4.3.13 *Case 8. Purely Imaginary Eigenvalues (Roots) $\lambda = \pm i\beta$: Centers*

Another example we study is the special system

$$\frac{dx}{dt} = -\beta y, \tag{56a}$$

$$\frac{dy}{dt} = -\beta x, \tag{56b}$$

which is just system (41a)–(41b) with $\alpha = 0$.

• SOLUTION. Let us first determine the eigenvalues before we rely on previously obtained results. To find the eigenvalues, we substitute

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{57}$$

and obtain

$$\begin{bmatrix} -\lambda & -\beta \\ \beta & -\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{58}$$

The eigenvalues satisfy the determinant condition

$$\lambda^2 + \beta^2 = 0. \tag{59}$$

The eigenvalues (roots) are purely imaginary $\lambda = \pm i\beta$, and then, by Euler's formula, the solution involves $\cos\beta t$ and $\sin\beta t$. To analyze solutions in the phase plane, we use previous results for polar coordinates and substitute $\alpha = 0$ into case 6 and 7. In this case from (44) and (46), we obtain

$$\frac{dr}{dt} = 0, \quad \frac{d\theta}{dt} = \beta. \tag{60}$$

Hence $r(t) = r(0)$ and $\theta(t) = \beta t + \theta(0)$. Since r is constant (but arbitrary), the trajectories are concentric circles about the origin. The circular orbits are counterclockwise if $\beta > 0$ (see figure 4.3.18a) and clockwise $\beta < 0$ (see figure 4.3.18b). Thus, the

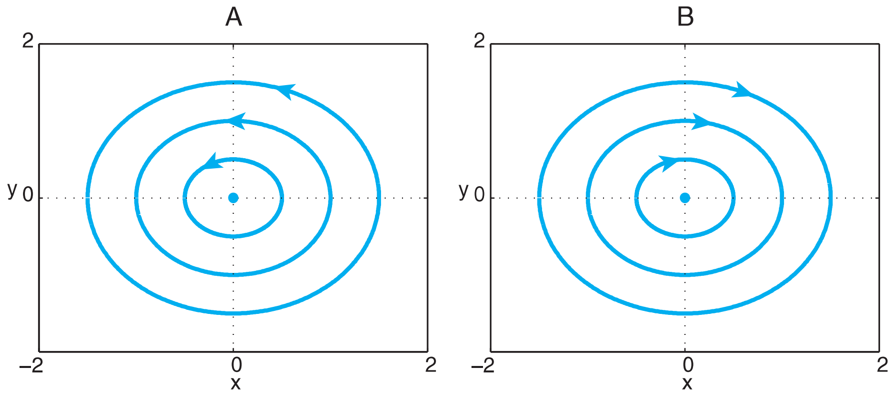

Figure 4.3.18 (a) Center: counterclockwise circles ($\beta > 0$), (b) center: clockwise circles ($\beta < 0$)

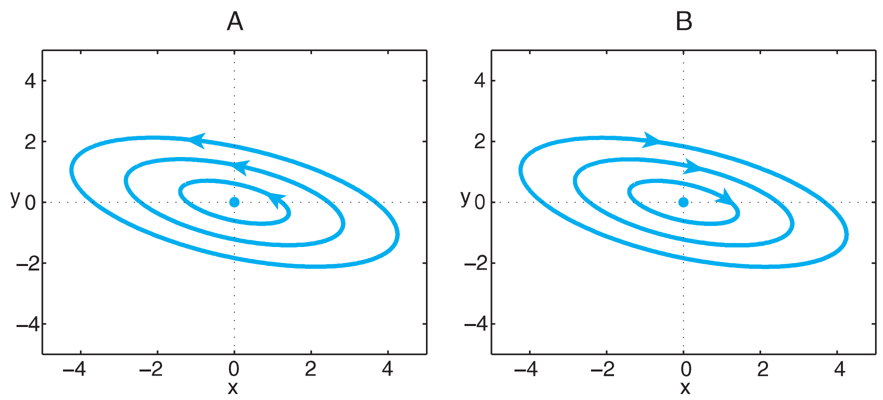

Figure 4.3.19 (a) Center: Counterclockwise skewed ellipses, (b) center: clockwise skewed ellipses.

motion is a periodic rotation around a circle centered at the origin. Appropriately, the origin is called a **center** and is a stable equilibrium. ♦

IN GENERAL, WHEN THE EIGENVALUES ARE PURELY IMAGINARY, THE ORIGIN IS A STABLE CENTER. The trajectories are "skewed ellipses" centered at the origin with axes of the ellipse not necessarily $x = 0$ and $y = 0$ as in our examples. Motion is periodic. The equilibrium is **stable** since nearby solutions do not move very far away. However, since the solutions do not approach the equilibrium, the equilibrium is not asymptotically stable. Typical phase diagrams are shown in Figure 4.3.19.

4.3.5 General Theorems

THEOREM 4.3.1 *Theorem on Phase Portraits and Stability of Linear Systems. We now summarize the phase plane behavior of a linear system of differential equations* *and the stability of the equilibrium at the origin:*

$$\frac{dx}{dt} = ax + by, \tag{61a}$$

$$\frac{dy}{dt} = cx + dy, \tag{61b}$$

in terms of the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{62}$$

The eigenvalues of the coefficient matrix satisfy the determinant condition, which is

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc) = 0. \tag{63}$$

Case 1: Two positive eigenvalues: unstable node (see figure 4.3.9)
Case 2: Two negative eigenvalues: stable node (see figure 4.3.11)
Case 3: One positive and one negative eigenvalue: unstable saddle point (see figure 4.3.13)
Case 4: Equal eigenvalues (repeated roots)
Case 5: One eigenvalue zero: stable if $\lambda_2 < 0$ (see figure 4.3.15) and unstable if $\lambda_2 > 0$
Case 6: Complex eigenvalues (positive real part): unstable spiral (see figure 4.3.16 a and b)
Case 7: Complex eigenvalues (negative real part): stable spiral (see figure 4.3.16 c and d)
Case 8: Complex eigenvalues (zero real part): stable center (see figure 4.3.19)

CLASSIFICATION OF STABILITY OF LINEAR SYSTEMS. Here we will classify the stability of the zero solution of linear systems in terms of the trace and determinant of the matrix. These results have nice extensions when we include the phase plane. The eigenvalues of the matrix solve the following quadratic equation (105) in term of the trace and determinant of the matrix:

$$\lambda^2 - \operatorname{tr}\mathbf{A}\lambda + \det\mathbf{A} = 0. \tag{64}$$

Using the quadratic formula, we have

$$\lambda = \frac{\operatorname{tr}\mathbf{A} \pm \sqrt{(\operatorname{tr}\mathbf{A})^2 - 4\det\mathbf{A}}}{2}. \tag{65}$$

If $4\det\mathbf{A} > (\operatorname{tr}\mathbf{A})^2$, then the eigenvalues are complex (the phase plane will be spirals), stable (spirals) if $\operatorname{tr}\mathbf{A} < 0$ and unstable (spirals) if $\operatorname{tr}\mathbf{A} > 0$. It is helpful to remember (108) and (109), so that $\lambda_1\lambda_2 = \det\mathbf{A}$ and $\lambda_1 + \lambda_2 = \operatorname{tr}\mathbf{A}$. If one eigenvalue is positive ($\lambda_1 > 0$) and the other negative ($\lambda_2 < 0$) (saddle points), the zero solution is automatically unstable, and this corresponds to $\det\mathbf{A} < 0$. The other regions will have real eigenvalues of the same sign (nodes) with $\det\mathbf{A} = \lambda_1\lambda_2 > 0$; the unstable case (nodes) satisfy $\lambda_1 + \lambda_2 = \operatorname{tr}\mathbf{A} > 0$ while the stable case (nodes) satisfy $\lambda_1 + \lambda_2 = \operatorname{tr}\mathbf{A} < 0$. This classification of the stability of the zero solution (including phase plane) for linear

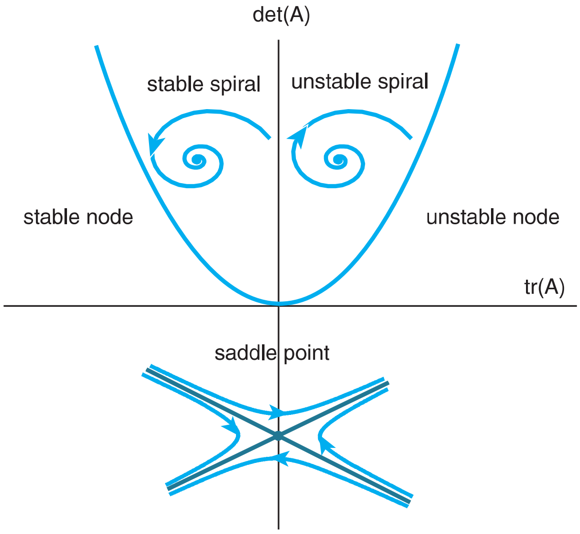

Figure 4.3.20 Phase plane and stability for linear systems classified by determinant and trace.

systems using trace and determinants is graphed in figure 4.3.20. From the figure, we see an interesting theorem: **the solution $\mathbf{x} = 0$ is stable for linear systems if and only if** $\det \mathbf{A} > 0$ **and** $\operatorname{tr} A < 0$. If $\det \mathbf{A} = 0$, at least one eigenvalue is zero.

Exercises

In Exercises 1–6, find the direction field using software,

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}, \tag{66}$$

where

1. $a = 0, b = 1, c = -4, d = 0.$
2. $a = 1, b = 3, c = 1, d = -1.$
3. $a = 2, b = 1, c = 1, d = 2.$
4. $a = -3, b = -2, c = 1, d = -5.$
5. $a = 3, b = -1, c = 1, d = 2.$
6. $a = -2, b = 0, c = 0, d = -3.$

In Exercises 7–22, determine the eigenvalues and eigenvectors if the eigenvalues are real (or use results from exercises from Section 4.2. if you have covered those exercises). Also classify the system (state whether stable or unstable node, stable or unstable spiral, center, saddle point) and in all cases sketch the phase plane of the linear system. (As a hint, problems with * have complex eigenvalues.) When checking your answers with those in the back of the book, keep in mind that any nonzero multiple of the given eigenvector may be used.

7. $\frac{dx}{dt}=x, \frac{dy}{dt}=x+2y.$

8. $\frac{dx}{dt}=2x-y, \frac{dy}{dt}=3x-2y.$

9*. $\frac{dx}{dt}=-x-5y, \frac{dy}{dt}=x+y.$

10. $\frac{dx}{dt}=2x-y, \frac{dy}{dt}=2x+5y.$

11*. $\frac{dx}{dt}=x-y, \frac{dy}{dt}=x+y.$

12*. $\frac{dx}{dt}=-2x+2y, \frac{dy}{dt}=-x.$

13. $\frac{dx}{dt}=-5x-4y, \frac{dy}{dt}=2x+y.$

14*. $\frac{dx}{dt}=x+5y, \frac{dy}{dt}=-2x-y.$

15. $\frac{dx}{dt}=y, \frac{dy}{dt}=2x+y.$

16*. $\frac{dx}{dt}=-x-2y, \frac{dy}{dt}=2x-y.$

17. $\frac{dx}{dt}=-5x-y, \frac{dy}{dt}=3x-y.$

18*. $\frac{dx}{dt}=x+2y, \frac{dy}{dt}=-4x-3y.$

19*. $\frac{dx}{dt}=-x+4y, \frac{dy}{dt}=-4x-y.$

20*. $\frac{dx}{dt}=3x+2y, \frac{dy}{dt}=-2x+3y.$

21. $\frac{dx}{dt}=4x+3y, \frac{dy}{dt}=3x+4y.$

22. $\frac{dx}{dt}=2x+3y, \frac{dy}{dt}=3x+2y.$

In Exercises 23–35, determine the eigenvalues and eigenvectors if the eigenvalues are real (or use results from exercises from Section 4.2 if you have covered those exercises), classify the system (state whether stable or unstable node, stable or unstable spiral, center, saddle point) and in all cases sketch the phase plane of the linear system. (As a hint, problems with * have complex eigenvalues.) When checking your answers with those in the back of the book, keep in mind that, any nonzero multiple of the given eigenvector may be used:

$$\frac{d\mathbf{x}}{dt}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{x}=\mathbf{A}\mathbf{x}, \tag{67}$$

where

23*. $a=0, b=1, c=-4, d=0.$

24. $a = 1, b = 3, c = 1, d = -1$.
25. $a = 2, b = 1, c = 1, d = 2$.
26*. $a = -3, b = -2, c = 1, d = -5$.
27*. $a = 3, b = -1, c = 1, d = 2$.
28. $a = -2, b = 0, c = 0, d = -3$.
29. $a = 1, b = 0, c = 1, d = -3$.
30. $a = -1, b = 3, c = 1, d = 1$.
31. $a = 4, b = -3, c = 1, d = 0$.
32*. $a = -1, b = 2, c = -2, d = -1$.
33. $a = 2, b = -1, c = 1, d = 0$.
34. $a = 3, b = 2, c = 0, d = 4$.
35. $a = 1, b = 0, c = 0, d = -3$.
36. For Exercises 23–28 without finding the eigenvalues, classify the system (stable or unstable node, stable or unstable spiral, center, saddle point), determine using the trace and determinant condition.
37. For Exercises 29–35 without finding the eigenvalues, classify the system (stable or unstable node, stable or unstable spiral, center, saddle point), determine using the trace and determinant condition.

In Exercises 38–41, graph the phase portrait given the eigenvalues and the eigenvectors:

38. $\lambda_1 = 1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
39. $\lambda_1 = 1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
40. $\lambda_1 = 1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
41. $\lambda_1 = -1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$