Linear Partial differential equations with Constant Co-efficients

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) Z = V(x,y) \longrightarrow (1)$$

As in the Case of ordinary differential equations the complete integral of eq. (1) Consists of the sum of two Parts:

(1) the most general integral of

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) Z = 0 \longrightarrow (2)$$

which is known as Complementary function (C.F)

(2) any Particular solution of eq. (1), which is known as Particular integral (P.I).

For Convenience $\frac{\partial}{\partial x} = D$

$$\frac{\partial}{\partial y} = D'$$

Homogeneous linear equation with Constant Co-efficients

t $f(D, D') = A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n \longrightarrow (3)$

where $A_0, A_1, A_2, \ldots, A_n$ are constants.

Then eq. (1) is known as homogeneous equation and takes the form

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \ldots + A_n D'^n) Z = V(x, y) \longrightarrow (4)$$

Complementary function

To find complementary function, consider

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \ldots + A_n D'^n) Z = 0 \longrightarrow (5)$$

let $Z = \phi(y + mx) \longrightarrow (6)$

be a solution of equation (5)

$$D^r Z = m^r \phi^{(r)}(y + mx)$$

$$D'^s Z = \phi^{(s)}(y + mx)$$

$$D^r D'^s Z = m^r \phi^{(r+s)}(y + mx)$$

Substituting in eq. (5)

$$(A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n)\,\phi^{(n)}(y+mx) = 0$$

which will be satisfied if

$$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0 \longrightarrow (7)$$

where $\phi^{(n)}(y+mx) \neq 0$

Eq. (7) is known as the auxiliary equation.

Let $m_1, m_2, \ldots, m_n$ be the roots of the eq. (7).

Three different cases arise:

Case 1:

$m_1, m_2, \ldots, m_n$ be distinct

C.F. corresponding to $m = m_1$ is

$$z = \phi_1(y + m_1 x)$$

where ϕ_1 is an arbitrary function

C.F. corresponding to $m = m_2$ is

$$z = \phi_2(y + m_2 x)$$

where ϕ_2 is an arbitrary function

and so on

The equation (5) being linear.
The sum of the solutions also a solution.
Hence

$$C.F. = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \cdots + \phi_n(y+m_nx) \longrightarrow (8)$$

Note: Eq. (7) is obtained by putting

$$D = m, \quad D' = 1$$

in (3), i.e. $f(m,1) = 0$.

Also, we can obtain auxiliary equation by putting $D = 1$, $D' = m$
i.e., $f(1,m) = 0$.

In this case

$$C.F. = \phi_1(x, m_1y) + \phi_2(x, m_2y) + \cdots + \phi_n(x, m_ny)$$

Example: Solve

$$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)Z = 0$$

Solution: auxiliary equation is

$$D = m \quad , \quad D' = 1$$

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$(m-1)(m^2 - 5m + 6) = 0$$

$$(m-1)(m-2)(m-3) = 0$$

$$m_1 = 1 \quad , \quad m_2 = 2 \quad , \quad m_3 = 3$$

$$C.F. = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x)$$

$$C.F. = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$$

Case 2: Imaginary roots

let a pair of complex roots of the eq. (7) be

$$u \pm iv$$

Then corresponding part of C.F. is

$$m_1 = u + iv \quad , \quad m_2 = u - iv$$

$$Z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$$

$$Z = \phi_1(y + ux + ivx) + \phi_2(y + ux - ivx)$$

$$Z = \phi_1(X + iY) + \phi_2(X - iY) \text{ , where}$$

$$X = y + ux \quad , \quad Y = vx$$

Let $\psi_1 = \phi_1 + \phi_2$, $i\psi_2 = \phi_1 - \phi_2$

Then $\phi_1 = \frac{1}{2}(\psi_1 + i\psi_2)$, $\phi_2 = \frac{1}{2}(\psi_1 - i\psi_2)$

$$Z = \frac{1}{2}\psi_1(x+iy) + \frac{1}{2}i\psi_2(x+iy) + \frac{1}{2}\psi_1(x-iy) - \frac{1}{2}i\psi_2(x-iy)$$

$$Z = \frac{1}{2}\{\psi_1(x+iy) + \psi_1(x-iy)\} + \frac{1}{2}i\{\psi_2(x+iy) - \psi_2(x-iy)\}$$

Example: Solve $(D^4 - D'^4)Z = 0$

Solution auxiliary equation is

$D = m$, $D' = 1$

A.E. is $m^4 - 1 = 0$

$(m^2-1)(m^2+1) = 0$

$(m-1)(m+1)(m^2+1) = 0$

$m_1 = 1$, $m_2 = -1$, $m_3 = i$, $m_4 = -i$

$$C.F. = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix).$$

Case 3: Repeated roots

Let two times repeated root of the eq. (7) be m.

Consider the equation

$$(D - mD')(D - mD')Z = 0 \longrightarrow (9)$$

Put

$$(D - mD')Z = u \longrightarrow (10)$$

Then $(D - mD')u = 0$

$$\frac{\partial u}{\partial x} - m\frac{\partial u}{\partial y} = 0$$

This equation is linear and has subsidiary equation

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{du}{0}$$

$$-m\,dx = dy$$

$$dy + m\,dx = 0$$

integrating

$$y + mx = \text{const}$$

$$du = 0$$

integrating $\quad u = \text{const}$

then $u = \phi(y+mx)$, where ϕ is an arbitrary function.

Substituting in eq. (10)

$$(D - mD')Z = \phi(y+mx)$$

$$\frac{\partial Z}{\partial x} - m\frac{\partial Z}{\partial y} = \phi(y+mx)$$

which has its subsidiary equations

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y+mx)}$$

$$-m\,dx = dy$$

$$dy + m\,dx = 0$$

integrating

$$y + mx = \text{Const}$$

$$\phi(y+mx)\,dx = dz$$

integrating

$$z = x\phi(y+mx) + \text{const}$$

then $z = x\phi(y+mx) + \psi(y+mx) \longrightarrow (11)$

where ψ is an arbitrary function.

Relation (11) is the Part of C.F. CorresPonding to the two times rePeated root m.

In general, if the root m is repeated 'r' times, the corresponding Part of C.F. is

$$z = x^{r-1}\phi_1(y+mx) + x^{r-2}\phi_2(y+mx) + \cdots + \phi_r(y+mx) \longrightarrow (12)$$

where $\phi_1, \phi_2, ---, \phi_r$ are arbitrary functions.

Note: If auxiliary equation is obtained by changing

$$D=1 \quad , \quad D'=m$$

the Part of C.F. corresponding to 'r' times repeated root 'm' is

$$Z = y^{r-1}\phi_1(x+my) + y^{r-2}\phi_2(x+my) + --- + \phi_r(x+my) \longrightarrow (13)$$

Example: Solve

$$(D^3 - 3D^2D' + 3DD'^2 - D'^3)Z = 0$$

Solution

$$D=m \quad , \quad D'=1$$

A.E. is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0$$

$$m_1=1, \quad m_2=1 \quad , \quad m_3=1$$

$$C.F. = x^2\phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+x)$$

Particular integral P.I.

For eq. ① $f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) Z = V(x,y)$

$$\frac{\partial}{\partial x} = D \quad , \quad \frac{\partial}{\partial y} = D'$$

$$f(D, D') Z = V(x,y)$$

The operator $\frac{1}{f(D,D')}$ is defined by an identity

$$f(D,D') \frac{1}{f(D,D')} V(x,y) = V(x,y)$$

Then P.I. is given by

$$\frac{1}{f(D,D')} V(x,y)$$

Example: solve $(D^2 - a^2 D'^2) Z = x$

Solution A.E. is

$$m^2 - a^2 = 0$$

$$(m-a)(m+a) = 0$$

$$m = a \quad , \quad m = -a$$

$$C.F. = \phi_1(y+ax) + \phi_2(y-ax)$$

$$P.I. = \frac{1}{D^2 - a^2 D'^2} x$$

$$= \frac{1}{D^2}\left(1 - a^2\frac{D'^2}{D^2}\right)^{-1} x$$

$$= \frac{1}{D^2}\left(1 + a^2\frac{D'^2}{D^2} + \cdots\right) x$$

$$= \frac{1}{D^2}(x + 0) = \frac{1}{D^2} x$$

$$= \frac{1}{D}\left(\frac{x^2}{2}\right) = \frac{x^3}{6}$$

or

$$P.I. = \frac{1}{D^2 - a^2 D'^2} x$$

$$= \frac{1}{-a^2 D'^2}\left(1 - \frac{D^2}{a^2 D'^2}\right)^{-1} x$$

$$= -\frac{1}{a^2 D'^2}\left(1 + \frac{D^2}{a^2 D'^2} + \cdots\right) x$$

$$= -\frac{1}{a^2 D'^2}(x + 0) = -\frac{1}{a^2 D'^2} x$$

$$= -\frac{1}{a^2} x \frac{y^2}{2}$$

$$= -\frac{x y^2}{2a^2}$$

Therefore complete integral is

$$Z = \phi_1(y + ax) + \phi_2(y - ax) + \frac{x^3}{6} \quad \text{or}$$

$$Z = \phi_1(y + ax) + \phi_2(y - ax) - \frac{1}{2a^2} x y^2$$

solve

$$(D^2 + 3DD' + 2D'^2) Z = x + y$$

$$P.I. = \frac{1}{(D^2 + 3DD' + 2D'^2)}(x + y)$$

$$= \frac{1}{F(D, D')}\phi^{(n)}(ax + by)$$

$$= \frac{1}{F(a, b)}\phi(ax + by)$$

let $u = x + y$

$(a = 1, b = 1)$

$$P.I. = \frac{1}{(1 + 3 + 2)}\iint u\, du$$

$$P.I. = \frac{1}{6}\int \frac{1}{2} u^2$$

$$P.I. = \frac{1}{6}\,\frac{1}{2}\,\frac{1}{3} u^3$$

$$P.I. = \frac{1}{36} u^3$$

$$= \frac{1}{36}(x + y)^3$$

بتطبيق النظرية نحصل
على هذا الحل

If $V(x,y)$ is a function of $(ax+by)$, we find that

Theorem: If $f(D,D')$ be of degree n, then

$$\frac{1}{f(D,D')}\phi^{(n)}(ax+by) = \frac{1}{f(a,b)}\phi(ax+by) \longrightarrow ①$$

Provided $f(a,b) \neq 0$

Proof: By direct differentiation

$$D^{r}\phi(ax+by) = a^{r}\phi^{(r)}(ax+by)$$

$$D'^{s}\phi(ax+by) = b^{s}\phi^{(s)}(ax+by)$$

$$D^{r}D'^{s}\phi(ax+by) = a^{r}b^{s}\phi^{(r+s)}(ax+by)$$

$$\therefore f(D,D')\phi(ax+by) = f(a,b)\phi^{(n)}(ax+by)$$

where $f(D,D')$ be of degree $\underline{\underline{n}}$

$$\frac{1}{f(D,D')}f(D,D')\phi(ax+by) = \frac{1}{f(D,D')}f(a,b)\phi^{(n)}(ax+by)$$

$$\phi(ax+by) = f(a,b)\frac{1}{f(D,D')}\phi^{(n)}(ax+by)$$

Dividing by $f(a,b) \neq 0$

$$\frac{1}{f(a,b)} \phi(ax+by) = \frac{1}{f(D,D')} \phi^{(n)}(ax+by)$$

$$\frac{1}{f(D,D')} \phi^{(n)}(ax+by) = \frac{1}{f(a,b)} \phi(ax+by)$$

Case when $f(a,b) = 0$

Let $f(D,D') = (bD-aD')^r g(D,D')$

where $g(D,D')$ is of degree $n-r$ and $g(a,b) \neq 0$

$$\frac{1}{f(D,D')} \phi^{(n)}(ax+by) = \frac{1}{(bD-aD')^r} \frac{1}{g(D,D')} \phi^{(n)}(ax+by)$$

$$= \frac{1}{(bD-aD')^r} \frac{1}{g(a,b)} \phi^{(r)}(ax+by)$$

$$= \frac{1}{g(a,b)} \frac{1}{(bD-aD')^r} \phi^{(r)}(ax+by) \longrightarrow (1)$$

Put $\frac{1}{bD-aD'} \phi^{(r)}(ax+by) = u$

$$\phi^{(r)}(ax+by) = (bD-aD')u$$

$$b\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y} = \phi^{(r)}(ax+by)$$

which has its subsidiary equations

$$\frac{dx}{b} = \frac{dy}{-a} = \frac{du}{\phi^{(r)}(ax+by)}$$

$$-a\,dx = b\,dy$$

$$a\,dx + b\,dy = 0$$

integrating

$$ax + by = \text{const}$$

$$\phi^{(r)}(ax+by)\,dx = b\,du$$

$$\phi^{(r)}(ax+by)\,x = bu$$

$$u = \frac{x}{b}\phi^{(r)}(ax+by)$$

$$\frac{1}{(bD-aD')^2}\phi^{(r)}(ax+by) = \frac{1}{bD-aD'}\,\frac{1}{bD-aD'}\,\phi^{(r)}(ax+by)$$

$$= \frac{1}{bD-aD'}\,u$$

$$= \frac{1}{bD-aD'}\,\frac{x}{b}\,\phi^{(r)}(ax+by)$$

$$= \frac{1}{bD}\left(1-\frac{a}{b}D'\right)^{-1}\frac{x}{b}\,\phi^{(r)}(ax+by)$$

$$= \frac{1}{bD}\left(1+\frac{a}{b}D'+\cdots\right)\frac{x}{b}\,\phi^{(r)}(ax+by)$$

$$= \frac{1}{bD}\left(\frac{x}{b}+0\right)\phi^{(r)}(ax+by)$$

$$= \frac{x^2}{2b^2} \phi^{(r)}(ax+by)$$

$$= \frac{x^2}{2!\,b^2} \phi^{(r)}(ax+by)$$

In general

$$\frac{1}{(bD-aD')^r} \phi^{(r)}(ax+by) = \frac{x^r}{r!\,b^r} \phi^{(r)}(ax+by)$$

substituting in 1

$$\frac{1}{f(D,D')} \phi^{(n)}(ax+by) = \frac{1}{g(a,b)} \frac{x^r}{r!} \frac{1}{b^r} \phi^{(r)}(ax+by).$$

Example: solve

$$(D^2+D'^2)Z = \cos mx \cos ny$$

Solution A.E. is

$$D=m,\ D'=1$$

$$m^2+1=0$$

$$m^2=-1$$

$$m=\pm i$$

$$C.F. = \phi_1(y+ix)+\phi_2(y-ix)$$

$$P.I. = \frac{1}{D^2+D'^2} \cos mx \cos ny$$

$$\cos mx \cos ny = \frac{1}{2}\{\cos(mx+ny)+\cos(mx-ny)\}$$

$$P.I. = \frac{1}{2}\frac{1}{D^2+D'^2}\{\cos(mx+ny)+\cos(mx-ny)\}$$

$$= \frac{-1}{2}\frac{1}{(m^2+n^2)}\{\cos(mx+ny)+\cos(mx-ny)\}$$

where we apply the Pove theorem

$a=m$, $b=n$ and $f(D,D')$ of degree 2

$$\phi^{(2)}(ax+by) = \cos(mx+ny)+\cos(mx-ny)$$

$$P.I. = -\frac{2}{2(m^2+n^2)}\cos mx \cos ny$$

$$P.I. = \frac{-1}{(m^2+n^2)}\cos mx \cos ny$$

$$\therefore Z = \phi_1(y+ix)+\phi_2(y-ix) - \frac{1}{m^2+n^2}\cos mx \cos ny$$

Solve $(D^2+DD'-2D'^2)Z = \sqrt{2x+y}$

$$P.I. = \frac{1}{(D^2+DD'-2D'^2)}(2x+y)^{1/2}$$

let $u^{1/2} = (2x+y)^{1/2}$ $\quad a=2$, $b=1$

$$P.I. = \frac{1}{F(D,D')}\phi^{(n)}(ax+by) = \frac{1}{F(a,b)}\phi(ax+by)$$

$$P.I. = \frac{1}{4+2-2}\iint u^{1/2}du = \frac{1}{4}\frac{2}{3}\int u^{3/2}du$$

$$= \frac{1}{6}\frac{2}{5}u^{5/2} = \frac{1}{15}u^{5/2} = \frac{1}{15}(2x+y)^{5/2}$$

If $V(x,y)$ involves $\sin(ax+by)$ or $\cos(ax+by)$, the method of undetermined co-efficients may be used to find particular integral.

Example: Solve

$$(D^2+DD'-6D'^2)Z = y\cos x \longrightarrow ①$$

Solution: A.E. is

$$m^2+m-6=0$$

$$(m+3)(m-2)=0$$

$$m=2,-3$$

$$C.F. = \phi_1(y+2x)+\phi_2(y-3x)$$

To find P.I., assume

$$Z = Ay\cos x + By\sin x + C\cos x + D\sin x \longrightarrow ②$$

Then

$$DZ = -Ay\sin x + By\cos x - C\sin x + D\cos x$$

$$D^2Z = -Ay\cos x - By\sin x - C\cos x - D\sin x$$

$$D'Z = A\cos x + B\sin x$$

$$D'^2Z = 0$$

$$DD'Z = -A\sin x + B\cos x$$

Substituting in eq. ①

$-Ay\cos x - By\sin x - C\cos x - D\sin x - A\sin x$
$+B\cos x = y\cos x$

$-Ay\cos x - By\sin x - (A+D)\sin x + (B-C)\cos x = y\cos x$

$\therefore A = -1 \;,\; B = 0$

$A + D = 0 \implies D = 1$

$B - C = 0 \implies C = 0$

Substituting in eq. ②

$$Z = -y\cos x + \sin x$$

Hence $P.I. = -y\cos x + \sin x$

$$\therefore Z = \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x.$$

solve

$$(D^2 - aD'^2)Z = x$$

$$\frac{1}{F(D,D')}\phi^{(n)}(ax+by) = \frac{1}{F(a,b)}\phi(ax+by)$$

$$\phi^{(n)}(ax+by) = x \qquad (a=1, b=0$$

$$P.I. = \frac{1}{(D^2 - a^2D'^2)}(x) = \frac{1}{1-a^2(0)}\iint x\,dx$$

$$= \int \frac{x^2}{2}\,dx = \frac{1}{2}\frac{1}{3}x^3$$

$$= \frac{1}{6}x^3$$