Third Year Stats. & Comp.
Stochastic Process
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Lecture # 7
First Entrance

Faculty of Science
Department of Mathematics

بسم الله الرحمن الرحيم

أبنائى طلاب الفرقة الثالثة - شعبة الاحصاء وعلوم الحاسب - كلية العلوم - جامعة دمياط

السلام عليكم ورحمة الله وبركاته

أرجو من الله العلى القدير أن يحفظكم برعايته وأن تكونوا جميعا بصحة جيدة أنتم وأسركم الكريمة، وأن تمر هذه المرحلة (المحنة) بأقل خسائرعلينا وعليكم وعلى مصرنا العزيزة وعلى شعوب العالم جميعا، كما أناشدكم جميعا أن تلتزموا بتوجيهات أولى الأمر!.

أرجو أن تستثمروا هذه الفترة فى الدراسة الجادة والبحث العلمى على الإنترنت لتعلم كل جديد عن مواضيع المقرر الدراسى"العمليات العشوائية" وأن تزيدوا من معلوماتكم وتنجزوا فى دراستكم قدر الإمكان.

بالنسبة للمحاضرات المتبقية فى مادة العمليات العشوائية: إن شاء الله سوف أرسل لكم دليل واضح لكل محاضرة متبقية، وذلك فى تاريخ وميعاد محاضرتنا الأسبوعية كما كان فى جدول الكلية **"الاحد من كل أسبوع"،**

ويتم ذلك عن طريق **د. إيمان الحديدى** المسئولة عن رفع المحاضرات على موقع الجامعة

1- أرجو أن تكون وصلتكم المحاضرة السادسة على موقع الجامعة فى ميعادها يوم السبت بتاريخ 2020/3/22 وهى إستكمال لآخر محاضرة توقفنا عندها!

2- مرفق لكم المحاضرة السابعة على موقع الجامعة فى ميعادها يوم السبت بتاريخ 2020/3/29 وهى استكمال للمحاضرة السادسة!

تنبيه هام: فى حالة وجود أى صعوبات تواجه (أى واحد فيكم) فى هذا المقرر، يجب طرحها ومناقشتها أولا مع باقى زملائكم، وان استمرت، ، تقوموا بإرسالها لى عن طريق د. غدير الشريف وان شاء اللة سوف أقوم بتذليلها والرد عليها.

حفظكم الله ووفقكم لما فيه رضاه

Prof. M A El-Shehawey

First-passage and first-return times

Def. (First-passage times)

The first-passage time T_{ij} is the RV that represents the first time to go from state i to state j, and is expressed as:

$$T_{ij} = \min\left\{k > 0 : X_k = j \middle| X_0 = i\right\}, \text{ for some } k = 1, 2, ...$$

k : is the number of transitions in a path from states i to j.

T_{ij} : is the minimum number of transitions in a path from i to j.

Mean Return Times

Another interesting random variable is the first return time. In particular, assuming the MC is in i, we consider the expected time (number of steps) needed until the chain returns to state i.

Def. (Time for first return or recurrence time)

We define the recurrence time (time for first visit or return time or waiting time) $T_i \equiv T_{ii}$ as the first time that the MC returns to state i:

$$T_{ii} = \min\left\{k > 0 : X_k = i \middle| X_0 = i\right\}.$$

The **probability** that the **first recurrence to state** i occurs at the n^{th}**-step** is

$$f_{ii}^{(n)} = \Pr\left\{T_{ii} = n\right\} = \Pr\left\{X_n = i, X_{n-1} \neq i, ..., X_1 \neq i \mid X_0 = i\right\}$$

$$= \Pr\left\{T_i = n \mid X_0 = i\right\}.$$

T_i: is the time for first visit to state i given $X_0 = i$.

The probability $f_{ii}^{(n)} \equiv f_i^{(n)}$ are known as first return probabilities to the state i occurs at the n^{th}-step.

Define $f_{ii}^{(0)} \equiv f_i^{(0)} = 0$. The probability $f_{ii}^{(n)} \equiv f_i^{(n)}$ is not the same as $p_{ii}^{(n)}$ which is the probability that a return occurs at the n^{th}-step, and includes possible returns at steps $1, 2, \ldots, n-1$ also

<u>Relationship between the probabilities</u> $p_{ii}^{(n)}$ and $f_{ii}^{(n)}$

There exist relationships between the n^{th}-step transition probabilities of a MC $p_{ii}^{(n)}$ and the first return probabilities $f_{ii}^{(n)}$.

The transition from state i to i at the n^{th}-step, $p_{ii}^{(n)}$, may have its first return to state i at any of the steps $j = 1, 2, \ldots, n$. It is easy to see that

$$p_{ii}^{(1)} (= p_{ii}) = f_i^{(1)},$$

$$p_{ii}^{(2)} = f_i^{(2)} + f_i^{(1)} p_{ii}^{(1)},$$

$$p_{ii}^{(3)} = f_i^{(3)} + f_i^{(1)} p_{ii}^{(2)} + f_i^{(2)} p_{ii}^{(1)},$$

this formula imply that the probability of a return at the third step $p_{ii}^{(3)}$ is the probability of a first return at the third step $f_i^{(3)}$, or the probability of a first return at the first step and a return two steps later $f_i^{(1)} p_{ii}^{(2)}$, or the probability of a first return at the second step and a return one step later $f_i^{(2)} p_{ii}^{(1)}$.

In general,

$$\begin{aligned} p_{ii}^{(n)} &= f_{ii}^{(0)} p_{ii}^{(n)} + f_{ii}^{(1)} p_{ii}^{(n-1)} + \ldots + f_{ii}^{(n)} p_{ii}^{(0)} \\ &= \sum_{r=1}^{n} f_{ii}^{(r)} p_{ii}^{(n-r)} = f_{ii}^{(n)} + \sum_{r=1}^{n-1} f_{ii}^{(r)} p_{ii}^{(n-r)}, \quad n \geq 2, \end{aligned}$$

since $f_{ii}^{(0)} = 0$ and $p_{ii}^{(0)} = 1$.

The above formulas become iterative formulas for the sequence of first returns $f_{ii}^{(n)} \equiv f_i^{(n)}$ which can be expressed as

$$f_i^{(1)} = p_{ii}^{(1)} \left(= p_{ii}\right),$$

$$f_i^{(n)} = p_{ii}^{(n)} - \sum_{r=1}^{n-1} f_i^{(r)} p_{ii}^{(n-r)}, \quad n \geq 2,$$

which implies that the probability that based on the condition that the MC started at i^{th} state at time $t = 0$, and would again be at i^{th} state at time $t = n$, provided it did not ever come to the i^{th} state at any of the times $t = 1, 2, ..., n-1$.

This is the **first return probability** for time $t = n$ (probability of first recurrence to i at the n^{th}-step, i.e., the probabilities that state i is revisited after the first, second, third, etc., transition times).

It is clear that we have a set of recurrence relations giving the recurrence time distribution $\left\{ f_{ii}^{(n)} \equiv f_i^{(n)} \right\}$ in terms of the $p_{ii}^{(n)}$.

Def. (Probability of ever returning to state)

We define the probability that the MC returns at least once (ever returning) to i (probability of recurrence to i) as $f_i \equiv f_{ii}$, where

$$f_i = f_{ii} = \Pr\left\{T_{ii} < \infty\right\} = \Pr\left\{T_i < \infty \mid X_0 = i\right\} = \sum_{n=1}^{\infty} f_{ii}^{(n)},$$

and probability of **never returning** to i is $1 - f_i = \Pr\left(T_{ii} = \infty\right)$.

Transience and recurrence

Def. (Transient state)

If $f_i = \Pr\left\{T_{ii} < \infty\right\} < 1$, then the state i is called **transient**, i.e., a state i is **transient** if the MC can leave but cannot return. In this case there is positive probability of never returning to state i.

Def. (Recurrent State)

If $f_i = \Pr\{T_{ii} < \infty\} = 1$, then the state i is called **recurrent** (or persistent), i.e., if once the MC reaches the state, it must return "or never leaves", the returns to the state i are sure events.

Lemma (1). The state j is **recurrent** or **transient** according as

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} = \infty \text{ or } < \infty.$$

Proof. By the first entrance theorem "with $p_{ii}^{(0)} = 1$"

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)} = \sum_{k=1}^{n} f_{ii}^{(k)} \sum_{n=k}^{\infty} p_{ii}^{(n-k)}$$

$$= f_{ii} \sum_{n=0}^{\infty} p_{ii}^{(n)} = f_{ii}\left(1 + \sum_{n=1}^{\infty} p_{ii}^{(n)}\right).$$

Hence, if $\sum_{n=1}^{\infty} p_{jj}^{(n)} < \infty$, we have $f_{ii} < 1$ and therefore state j is **transient**. Now

$$\sum_{n=1}^{N} p_{ii}^{(n)} = \sum_{n=1}^{N}\sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)} = \sum_{k=1}^{N} f_{ii}^{(k)} \sum_{n=k}^{N} p_{ii}^{(n-k)} \le \sum_{k=1}^{N} f_{ii}^{(k)} \sum_{u=0}^{N} p_{ii}^{(u)},$$

then $f_{ii} = \sum_{k=1}^{\infty} f_{ii}^{(k)} \ge \sum_{k=1}^{N} f_{ii}^{(k)} \ge \dfrac{\sum_{n=1}^{N} p_{ii}^{(n)}}{\sum_{u=0}^{N} p_{ii}^{(u)}} \to 1$ as $N \to \infty$, therefore

$\sum_{n=1}^{\infty} p_{jj}^{(n)} = \infty$ implies that $f_{ii} = 1$, that is state j is **recurrent**.

Def. (Probability of first passage at time n)

The probability of the first-passage (first visit or reaching) to state j given $X_0 = i$ occurs at the n^{th}-**step** is

$$f_{ij}^{(n)} = \Pr\{X_n = j, X_{n-1} \ne j, \ldots, X_1 \ne j \mid X_0 = i\}, \text{ for } n = 1, 2, \ldots.$$

Then the probability distribution of the first passage time T_{ij} is

$$f_{ij}^{(n)} = \Pr\{T_{ij} = n\} = \Pr\{T_j = n \mid X_0 = i\}, \text{ for } n = 1, 2, \ldots,$$

where $T_j = \min\{k > 0 : X_k = j\}$ is the unconditional first passage time to state j, i.e., T_j is a stochastic variable denoting the first time that the chain enters state j.

<u>**Def**. (Probability of first-passage or of ever hits state)</u>

The **probability** of the first-passage (ever reaching) to state j starting from the state i is

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} = \Pr\{T_{ij} < \infty\} = \Pr\left(T_j < \infty \mid X_0 = i\right)$$

$$= \Pr\left(\text{ever visits } j \text{ starting from } i\right),$$

whence $f_{ij}^{*} = 1 - f_{ij} = \Pr\left(T_{ij} = \infty\right) = \Pr\left(X_n \neq j \quad \text{for all } m \geq 1 \mid X_0 = i\right)$.

We have for $n = 1$, $\{T_{ij} = 1\} = \{X_1 = j \mid X_0 = i\}$

$$\rightarrow f_{ij}^{(1)} = \Pr\left(T_{ij} = 1\right) = \Pr\left(X_1 = j \mid X_0 = i\right) = p_{ij}.$$

For $n > 1$, $\{T_{ij} = n\} = \{X_n = j, X_m \neq j \quad \text{for } 1 \leq m \leq n-1 \mid X_0 = i\}$.

For example, if $j = 3$ and

$X_0 = 4, X_1 = 2, X_2 = 2, X_3 = 5, X_4 = 3, X_5 = 1, X_6 = 3, \cdots$, then $T_3 = 4$.

In the following theorem we introduce, for $n > 1$, the first passage time is n if the first transition is from state i to some state $k\ (k \neq j)$ and then the first passage time from state k to state j is $n-1$.

Theorem (1). (First step-decomposition or Iterative relations)

The first passage time probability from i to j in n steps, $f_{ij}^{(n)}$, can be determined iteratively by

$$f_{ij}^{(n)} = \begin{cases} p_{ij} & \text{if } \; n=1 \\ \sum_{k;k\neq j} p_{ik} f_{kj}^{(n-1)} & \text{if } \; n\geq 2 \end{cases} \qquad (1)$$

Proof. This expression follows, since conditioning on X_1 and using the Markov property. Suppose $n=1$, the definition yields

$$f_{ij}^{(1)} = \Pr\left(T_{ij}=1\right) = \Pr\left(X_1 = j \middle| X_0 = i\right) = p_{ij}.$$

For $n \geq 2$, $f_{ij}^{(n)} = \Pr\left(T_{ij}=n\right) = \Pr\left(X_n = j, X_m \neq j \text{ for } 1\leq m\leq n-1 \,|\, X_0 = i\right)$

$$= \sum_{k\in SS, k\neq j} \Pr\left(X_n = j, X_m \neq j \text{ for } 2\leq m\leq n-1, X_1 = k \,|\, X_0 = i\right)$$

$$= \sum_{k\in SS, k\neq j} \Pr\left(X_n = j, X_m \neq j \text{ for } 2\leq m\leq n-1 \,|\, X_1 = k, X_0 = i\right)\Pr\left(X_1 = k \,|\, X_0 = i\right)$$

$$= \sum_{k\in SS, k\neq j} \Pr\left(X_n = j, X_m \neq j \text{ for } 2\leq m\leq n-1 \,|\, X_1 = k\right)\Pr\left(X_1 = k \,|\, X_0 = i\right)$$

$$= \sum_{k\in SS, k\neq j} \Pr\left(X_n = j, X_m \neq j \text{ for } 1\leq m\leq n-2 \,|\, X_0 = k\right)\Pr\left(X_1 = k \,|\, X_0 = i\right).$$

Thus, the result follows.

Summing over all n in (1), yields the linear equations:

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} = p_{ij} + \sum_{k;k\neq j} p_{ik} f_{kj}, \text{ for } i \in SS,$$

for the passage or hitting probabilities from states i to j.

Theorem (2). (First entrance theorem)

For any two states i and j in a MC $\{X_n, n = 0,1,2,...\}$, the relation of probability $p_{ij}^{(n)}$ in terms of $f_{ij}^{(n)}$ is given by

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}, \text{ for } n = 1, 2, \ldots .$$

Proof. $p_{ij}^{(n)} = \Pr\left(X_n = j \mid X_0 = i\right)$

$$= \sum_{m=1}^{n} \Pr\left(X_n = j, X_m = j, X_{k-1} \neq j \text{ for } 1 \leq k \leq m-1 \mid X_0 = i\right)$$

$$= \sum_{m=1}^{n} \Pr\left(X_n = j \mid X_m = j, X_{k-1} \neq j \text{ for} 1 \leq k \leq m-1, X_0 = i\right)$$

$$\Pr\left(X_m = j, X_{k-1} \neq j \text{ for } 1 \leq k \leq m-1, X_0 = i\right)$$

$$= \sum_{m=1}^{n} \Pr\left(X_n = j \mid X_m = j\right) \Pr\left(X_m = j, X_{k-1} \neq j \text{ for } 1 \leq k \leq m-1, X_0 = i\right)$$

$$= \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}, \text{ for } \; n = 1, 2, \ldots$$

Def. <u>(Hitting times)</u>

The hitting time to state j at time n starting from i is defined by

$$h_{ij} = E\left[T_{ij}\right] = E\left[T_j \mid X_0 = i\right] = \sum_{n=0}^{\infty} n\, f_{ij}^{(n)}, \; T_{ij} < \infty .$$

Some-times called the <u>mean first passage time</u> or the expected number of steps needed to go from start i ends up on first reaching state j in a finite number of steps n).

The <u>return time</u> or <u>mean recurrence time</u> is the expected number of steps to return to state i starting from state i, for the first time:

$$h_{ii} \equiv h_i = \sum_{n \geq 1} n \Pr\left(X_n = i \middle| X_0 = i, X_m \neq i, m < n\right).$$

Def. <u>(Mean recurrence time)</u>

The mean recurrence time of state i is the expected return time to state i: $h_i = E\left[T_{ii}\right] = E\left[T_i \mid X_0 = i\right] = \sum_{k=1}^{\infty} k \Pr\left\{T_{ii} = k\right\} = \sum_{n=0}^{\infty} n\, f_{ii}^{(n)}$.

<u>A simple way to calculate</u> h_{ij} & h_i

For a fixed j, $\left\{h_{ij},\ i \in T\right\}$ satisfy the set of linear simultaneous equations (i.e., given $\left\{p_{ij}\right\}$, we can obtain $\left\{h_{ij},\ i \in T\right\}$ by solving the following equations):

$$h_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}\left(1 + h_{kj}\right),$$

where $h_j = E\left[T_j \mid X_0 = j\right] = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$ the mean recurrent time of state j.

Since $p_{ij} + \sum_{k \neq j} p_{ik} = 1$, then

$$h_{ij} = 1 + \sum_{k \neq j} p_{ik} h_{kj}, \text{ for } i \neq j$$

$$h_j \equiv h_{jj} = 1 + \sum_{k \neq j} p_{jk} h_{kj}.$$

Even though state j is **recurrent**, it is not necessary that h_j is <u>finite</u> when the state space is infinite, i.e., If $f_{ij} < 1$, we have $h_{ij} = \infty$.

<u>Recurrent states are classified into two types</u>, it is called

- **null recurrent** if h_i is infinite: $h_i = E\left[T_{ii}\right] = \infty$, or
- **positive recurrent** (non-null), if h_i is finite: $h_i = E\left[T_{ii}\right] < \infty$.

An **absorbing** state is a special kind of positive recurrent state.

<u>Example (1)</u>. Consider the MC $\left\{X_n, n = 0,1,...\right\}$ on the state space $SS = \left\{0,1,2,3\right\}$ with the TPM:

$$\mathbf{M} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Classify the states of the MC$\{X_n, n = 0,1,...\}$. Is the state 0 recurrent or transient? If it is recurrent, compute the mean recurrence time. What about the state 2?

<u>Solution</u>. There are three classes:$\{0,1\},\{2\}$ and $\{3\}$.

The **probability of first return** to state 0 will occur at time n, when the initial state is 0:

$$f_{00}^{(n)} = \Pr\left(X_n = 0, X_m \neq 0 \text{ for } 1 \le m \le n-1 \mid X_0 = 0\right) \text{ for } n = 1,2,\cdots,$$

are obtained by

$$f_{00}^{(1)} = \Pr\left(X_1 = 0 \mid X_0 = 0\right) = p_{00} = 1/2$$

$$f_{00}^{(2)} = \Pr\left(X_2 = 0, X_1 \neq 0 \mid X_0 = 0\right) = p_{01}p_{10} = (1/2)(1/3)$$

$$\vdots$$

$$f_{00}^{(n)} = \frac{1}{2}\left(\frac{2}{3}\right)^{n-2}\left(\frac{1}{3}\right), \quad n \ge 2.$$

Therefore, $f_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)} = \frac{1}{2} + \sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{2}{3}\right)^{n-2}\left(\frac{1}{3}\right) = \frac{1}{2} + \frac{1}{6}\sum_{n=2}^{\infty}\left(\frac{2}{3}\right)^{n-2} = 1.$

Hence, the state 0 is recurrent.

The mean recurrence time of state 0 is calculated by

$$h_0 = E\left[T_0 \mid X_0 = 0\right] = \sum_{n=1}^{\infty} n f_{00}^{(n)} = \frac{1}{2} + \frac{1}{6}\sum_{n=2}^{\infty} n\left(\frac{2}{3}\right)^{n-2} = \frac{1}{2} + \frac{1}{4}\sum_{n=2}^{\infty} n\left(\frac{2}{3}\right)^{n-1}$$

$$= \frac{1}{2} + \frac{1}{4}\left[\frac{2(2/3) - (2/3)^2}{(1-2/3)^2}\right] = \frac{1}{2} + \frac{1}{4}(8) = \frac{5}{2}.$$

So, state 0 is positive recurrent.

Note that. The mean recurrence time of state 0 can be also obtained by conditioning on the first state visited.

$$h_0 = \sum_k p_{0k} h_{k0} = p_{00} h_{00} + p_{01} h_{10} = p_{00} E[T_0 \mid X_1 = 0] + p_{01} E[T_0 \mid X_1 = 1]$$

$$= \frac{1}{2} E[T_0 \mid X_1 = 0] + \frac{1}{2} E[T_0 \mid X_1 = 1] = \frac{1}{2} + \frac{1}{2}\{1 + E[T_{10}]\} = 1 + \frac{1}{2} E[T_{10}].$$

Since T_{10} follows geometric distribution with mean 3 we obtain the same result.

Similarly for the state 2, The **probability of first return** to state 2 will occur at time n, when the initial state is 2:

$$f_{22}^{(n)} = \Pr\left(X_n = 2, X_m \neq 2 \text{ for } 1 \leq m \leq n-1 \mid X_0 = 2\right), \text{ for } n = 1, 2, \cdots$$

are obtained by

$$f_{22}^{(1)} = \Pr\left(X_1 = 2 \mid X_0 = 2\right) = p_{22} = 1/4,$$

$$f_{22}^{(2)} = \Pr\left(X_2 = 2, X_1 \neq 2 \mid X_0 = 2\right) = p_{21} p_{12} = (1/4)(0) = 0,$$

$$f_{00}^{(n)} = 0, \text{ for } n \geq 2.$$

Therefore, $f_{22} = \sum_{n=1}^{\infty} f_{22}^{(n)} = \frac{1}{4} + 0 = \frac{1}{4}$. So, the state 2 is transient.

<u>**Example (2)**.(Gambler's Ruin Problem)</u>

A gambler keeps playing a game until his or her fortune reaches to 0 or L. It is assumed that at each play of the game the gambler wins one unit with prob. p and loses one unit with prob. $q = 1 - p$.

1- What is the probability that, starting with i units, the gambler's fortune will reach L before going down to 0.
2- What is the expected number of bets until the gambler's fortune will reaches 0 or L, starting in i units. Assume that $p \neq 1/2$

Solution. Let X_n be the gambler's fortune after n^{th}-play. Then X_n takes states of the state space $\{0, 1, \ldots, L\}$ and $\{X_n, n = 0, 1, \ldots\}$ will be a MC with the following transition probabilities:

$$p_{i,i+1} = p,\ p_{i,i-1} = 1 - p = q, \text{ for } i = 1,2,\ldots,L-1; \text{ and } p_{00} = p_{LL} = 1.$$

So, the states are classified into three classes: $\{0\},\{1,2,\ldots,L-1\},\{L\}$.

Here the state 0 and the state L are **absorbing** and **recurrent**, but the states in the class of $\{1,2,\ldots,L-1\}$ will be **transient**. In the long-run $(n \to \infty)$, therefore, X_n goes to state 0 or state L.

We are interested in obtaining f_{iL}, $i \in \{0,1,\ldots,L\}$, the probability the state eventually goes to state L starting in i. obviously,

$$f_{0,L} = 0 \text{ and } f_{L,L} = 1.$$

The following holds from theorem (1):

$$\begin{aligned} f_{1L} &= pf_{2L}, \\ f_{iL} &= pf_{i+1L} + qf_{i-1L}, \text{ for } i = 2,\ldots,L-2 \\ f_{L-1L} &= qf_{L-2L} + p. \end{aligned}$$

It turns out that $f_{iL} = pf_{i+1L} + qf_{i-1L}$, for $i = 1,2,\ldots,L-1$.

Rewriting this equation yields

$$f_{i+1L} - f_{iL} = \frac{q}{p}\left(f_{iL} - f_{i-1L}\right), \text{ for } i = 1,2,\ldots,L-1.$$

Therefore, we have

$$f_{iL} - f_{i-1L} = \frac{q}{p}\left(f_{i-1L} - f_{i-2L}\right) = \ldots = \left(\frac{q}{p}\right)^{i-1} f_{1L}, \text{ for } i = 2,3,\ldots,L-1$$

$$\sum_{k=2}^{i}\left(f_{kL} - f_{k-1L}\right) = f_{1L}\sum_{k=2}^{i}\left(\frac{q}{p}\right)^{k-1}.$$

Hence,

$$f_{iL} = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)} f_{1L}, & \text{if } q/p \neq 1 \\ if_{1L}, & \text{if } q/p = 1 \end{cases},$$

f_{1L} is obtained from the fact that, $f_{LL} = 1$. So, finally we have

$$f_{iL} = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^L}, & \text{if } q/p \neq 1 \\ i/L, & \text{if } q/p = 1 \end{cases}.$$

Remarks

1-For a large L $(L \to \infty)$ the above becomes

$$f_{iL} \to \begin{cases} 1-(q/p)^i, & \text{if } p > 1/2 \\ 0, & \text{if } p \leq 1/2 \end{cases}$$

2-The probability that, starting in state i, state j $(j = 1, 2, \ldots, L)$ is eventually reached before state 0 is given by

$$f_{ij} = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^j}, & \text{if } q/p \neq 1 \\ i/j, & \text{if } q/p = 1 \end{cases}$$

Let X_j be the winning on the j^{th} bet and B be the number of bets until the gambler's fortune reaches 0 or L. Then,

$$B = \min\left\{ m : \sum_{j=1}^{m} X_j = -i \text{ or } L-i \right\}.$$

Note that B is a stopping time for the X_j's'. So,

$$E\left[\sum_{j=1}^{B} X_j\right] = E[B]E\left[X_j\right] = (2p-1)E[B].$$

On the other hand, it follows that $\sum_{j=1}^{B} X_j = \begin{cases} L-i & \text{w.p. } \alpha \\ -i & \text{w.p. } 1-\alpha \end{cases},$

where α is the probability that the fortune reaches L before 0, that

$$\alpha = \frac{1-(q/p)^i}{1-(q/p)^L}.$$

Therefore, $E\left[\sum_{j=1}^{B} X_j\right] = (L-i)\alpha - i(1-\alpha) = L\alpha - i$. So,

$$(2p-1)E[B] = L\alpha - i, \text{ or } E[B] = \frac{L\alpha - i}{2p-1} = \frac{1}{2p-1}\left\{\frac{L\left(1-(q/p)^i\right)}{1-(q/p)^L} - i\right\}.$$

<u>Example (3)</u>.

Consider a MC with state space $SS = \{1,2,3\}$ and TPM $\mathbf{M}$:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{3}{5} & \frac{1}{15} \end{pmatrix}.$$

Determine the probability that the first visit to state $j = 3$ will occur at time n, when the initial state is $i \in SS$.

What is the probability that state $j = 3$ is never visited?

Solution. Let $f_{ij}^{(n)}$ the first passage time probabilities to state j starting from state i, then $f_{ij}^{(n)} = \Pr\left(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i\right)$,

$$f_{ij}^{(n)} = \begin{cases} p_{ij}, & n = 1 \\ \sum_{k \varepsilon SS - \{j\}} p_{ik} f_{kj}^{(n-1)}, & n \geq 2 \end{cases}.$$

In a **matrix form**, suppose that the column vector of the n-<u>step first passage time probabilities</u> to the target state $j = 3$ is **desired**:

$$\mathbf{f}^{(n)} = \mathbb{Z}\mathbf{f}^{(n-1)}, \text{ with, } \mathbf{f}^{(n)} = \left(f_{ij}^{(n)}\right)_{i,j\in SS} = \left(\mathbf{f}_j^{(n)}\right)_{j\in SS} = \left(\mathbf{f}_1^{(n)} \quad \mathbf{f}_2^{(n)} \quad \mathbf{f}_3^{(n)}\right):$$

$$\mathbf{f}_3^{(n)} = \left(f_{i3}^{(n)}\right)_{i\in SS} = \begin{pmatrix} f_{13}^{(n)} \\ f_{23}^{(n)} \\ f_{33}^{(n)} \end{pmatrix},$$

and the matrix $\mathbb{Z}$ is the matrix $\mathbf{M}$ with column j of the target state replaced by a column of **zeroes**:

$$\mathbb{Z} = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\begin{pmatrix} p_{11} & p_{12} & 0 \\ p_{21} & p_{22} & 0 \\ p_{31} & p_{32} & 0 \end{pmatrix}.$$

When $j=3$, i.e., to compute $\mathbf{f}_3^{(n)}$, one may start with $n=1$.

The column probability vector of going from any state $i \in SS$ to state $j=3$ for the first time in one step, $\mathbf{f}_3^{(1)}$ is simply the vector of one-step TP of the third column $(p_{i3})_{i \in SS}$ of the TPM $\mathbf{M}$, that is

$$\mathbf{f}^{(1)} \equiv \mathbf{f}_3^{(1)} = \left(f_{i3}^{(1)}\right)_{i \in SS} = \begin{pmatrix} f_{13}^{(1)} \\ f_{23}^{(1)} \\ f_{33}^{(1)} \end{pmatrix} = \begin{pmatrix} p_{13} \\ p_{23} \\ p_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{15} \end{pmatrix},$$ with the matrix $\mathbb{Z}$ is

$$\mathbb{Z} = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{3}{5} & 0 \end{pmatrix}.$$ Then $$\mathbf{f}^{(2)} = \mathbb{Z}\mathbf{f}^{(1)} = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{3}{5} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{15} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{18} \\ \frac{1}{5} \end{pmatrix},$$

$$\mathbf{f}^{(3)} = \mathbb{Z}\mathbf{f}^{(2)} = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{3}{5} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{18} \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{108} \\ \frac{1}{30} \end{pmatrix},$$

$$\mathbf{f}^{(4)} = \mathbb{Z}\mathbf{f}^{(3)} = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{3}{5} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{108} \\ \frac{1}{30} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{648} \\ \frac{1}{180} \end{pmatrix}, \cdots.$$ Thus,

$$f_{13}^{(n)} = 0, \quad f_{23}^{(n)} = \frac{1}{3}\left(\frac{1}{6}\right)^{n-1}, \; n = 1, 2, \ldots, \quad f_{33}^{(n)} = \begin{cases} \frac{1}{15}, & n = 1 \\ \frac{1}{5}\left(\frac{1}{6}\right)^{n-2}, & n = 2, 3, \ldots \end{cases}.$$

Therefore, the first passage probabilities to state 3 starting from state $i=1$ is $f_{13} = \sum_{n \geq 1} f_{13}^{(n)} = 0$, the first passage probabilities to state 3 starting from state $i=2$ is

$$f_{23}=\sum_{n\geq 1} f_{23}^{(n)}=\frac{1}{3}\sum_{n\geq 1}\left(\frac{1}{6}\right)^{n-1}=\frac{1}{3}\frac{1}{1-1/6}=\frac{1}{3}\times\frac{6}{5}=\frac{2}{5},$$

the first passage probabilities to state 3 starting from state $i=3$ is

$$f_{33}=\sum_{n\geq 1} f_{33}^{(n)}=\frac{1}{15}+\frac{1}{5}\sum_{n\geq 2}\left(\frac{1}{6}\right)^{n-2}=\frac{1}{15}+\frac{1}{5}\frac{1}{1-1/6}=\frac{1}{15}+\frac{1}{5}\times\frac{6}{5}=\frac{23}{75}.$$

Therefore, the vector of first passage probabilities to state 3 starting from $i\in SS$ is $\mathbf{f}_3=\left(f_{i3}\right)_{i\in SS}=\begin{pmatrix} f_{13}\\ f_{23}\\ f_{33}\end{pmatrix}=\begin{pmatrix} 0\\ 2/5\\ 23/75\end{pmatrix}$, and the vector of the probability that state 3 is never visited starting from $i\in SS$ is

$$\mathbf{f}_3^*=\left(1-f_{i3}\right)_{i\in SS}=\begin{pmatrix} 1-f_{13}\\ 1-f_{23}\\ 1-f_{33}\end{pmatrix}=\begin{pmatrix} 1\\ 3/5\\ 52/75\end{pmatrix}.$$