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Lecture # 7
Combinatorics
Stirling Numbers

Faculty of Science
Department of Mathematics

بسم الله الرحمن الرحيم

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السلام عليكم ورحمة الله وبركاته

أرجو من الله العلى القدير أن يحفظكم برعايته وأن تكونوا جميعا بصحة جيدة أنتم وأسركم الكريمة، وأن تمر هذه المرحلة (المحنة) بأقل خسائرعلينا وعليكم وعلى مصرنا العزيزة وعلى شعوب العالم جميعا، كما أناشدكم جميعا أن تلتزموا بتوجيهات أولى الأمر!.

أرجو أن تستثمروا هذه الفترة فى الدراسة الجادة والبحث العلمى على الإنترنت لتعلم كل جديد عن مواضيع مقرر التآلفيات التى تناولناها والمتبقية وأن تزيدوا من معلوماتكم وتنجزوا فى دراستكم قدر الإمكان.

بالنسبة للمحاضرات المتبقية فى مادة التآلفيات: إن شاء الله سوف أرسل لكم دليل واضح لكل محاضرة متبقية، وذلك فى تاريخ وميعاد محاضرتنا الأسبوعية كما كان فى جدول الكلية **"السبت من كل أسبوع"،**
ويتم ذلك عن طريق **د. إيمان الحديدى** المسئولة عن رفع المحاضرات على موقع الجامعة

1- أرجو أن تكون وصلتكم المحاضرة السادسة على موقع الجامعة فى ميعادها يوم السبت بتاريخ 2020/3/21 وهى إستكمال لآخر محاضرة توقفنا عندها!

2- مرفق لكم المحاضرة السابعة على موقع الجامعة فى ميعادها يوم السبت بتاريخ 2020/3/28 وهى استكمال للمحاضرة السادسة!

تنبيه هام: فى حالة وجود أى صعوبات تواجه (أى واحد فيكم) فى هذا المقرر، يجب طرحها ومناقشتها أولا مع باقى زملائكم، وان استمرت، تقوموا بإرسالها لى عن طريق د. غدير الشريف وان شاء اللة سوف أقوم بتذليلها والرد عليها.

حفظكم الله ووفقكم لما فيه رضاه

Prof. M A El-Shehawey

Stirling Numbers

We focus here on Stirling numbers which arise in a variety of combinatorics problems. They are named after James Stirling (1692-1770), who introduced them in the 18th century. Two different sets of numbers bear this name: the Stirling numbers of the first kind and the Stirling numbers of the second kind, for integers n and k with $0 \le k \le n$, are written as

second kind $S(n,k) \equiv \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, first kind $\left[\begin{matrix} n \\ k \end{matrix} \right] = (-1)^{n-k} s(n,k)$,

for the Stirling numbers of the second and first kind, respectively.

The Stirling numbers of the first and second kind can be understood to be inverses of one-another.

We begin with $\{S(n,k)\}_{k=0}^{n}$, where $S(n,k) \equiv \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is a Stirling number of the second kind.

1- Stirling Numbers of the Second Kind

Def (1). (Partitions of Set)

Let S be a nonempty set, a partition $\{A_1, A_2, A_3, \ldots, A_k\}$ of size k, is called a partition of the set S, if the following conditions are hold:

1- $\phi \ne A_i \subseteq S \;\; \forall \; 1 \le i \le k$ (Inclusion)

2- $A_i \cap A_j = \phi, \text{ for } \; 1 \le i \ne j \le k$ (Disjoint)

3- $S = A_1 \cup A_2 \cup \ldots \cup A_k$ (Exhaustive)

Combinatorial meaning for $S(n,k)$: Let S be a nonempty set. A set partition of S is a collection of pairwise disjoint non-empty subsets of S whose union is S.

For example, let $S = \{a,b,c\}$. There are 5 set **partitions** of S, namely $S = \{a,b,c\}$ itself, the set partition consisting of **three** subsets, i.e.$\{\{a\},\{b\},\{c\}\}$, and the **three** set partitions consisting of two subsets, i.e. $\{\{\{a\},\{b,c\}\},\{\{b\},\{a,c\}\},\{\{c\},\{a,b\}\}\}$.

Def (1) (Stirling numbers of the second kind).

A partition of a set $[n] \equiv \{1,2,3,\ldots,n\}$ is an equivalence relation on that set. The equivalence classes are called parts. $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the number of ways of partitioning the set $[n]$ into k non-empty subsets or parts, i.e., the Stirling numbers of the second kind $S(n,k)$ (with a capital "S") "are also called Stirling subset numbers" count the number of ways to partition a set of n elements into k non-empty subsets. Equivalently, it is the number of ways that n distinguishable balls can be placed into k indistinguishable cells, with no cell empty. By definition $S(n,k) = 0$ if $k = 0$ or $k > n$. For technical reasons we define $S(0,0) = 1$. The following is a table (1) of values for the Stirling numbers of the second kind $S(n,k) \equiv \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ where $0 \le k \le n$ and $0 \le n \ge 10$:

	$S(n,k)=\left\{ {n \atop k} \right\}$										
k / n	0	1	2	3	4	5	6	7	8	9	10
0	1	0									
1	0	1									
2	0	1	1								
3	0	1	3	1							
4	0	1	7	6	1						
5	0	1	15	25	10	1					
6	0	1	31	90	65	15	1				
7	0	1	63	301	350	140	21	1			
8	0	1	127	966	1701	1050	266	28	1		
9	0	1	255	3025	7770	6951	2646	462	36	1	
10	0	1	511	9330	34105	42525	22827	5880	750	45	1

Table (1): Stirling numbers of the second kind

For instance, the number 25 in column $k=3$ and row $n=5$ is given by $25=7+(3\times 6)$, where 7 is the number above and to the left of 25, 6 is the number above 25 and 3 is the column containing the 6.

One style we see stand out from the first rows of this matrix is:

$$S(n,2)\equiv\left\{ {n \atop 2} \right\}=2^{n-1}-1=\frac{2^n-2}{2}, \text{ for } n\geq 2.$$

Note that $2^{n-2}-2$ is the number of non-empty proper subsets of $[n]\equiv\{1,2,3,\ldots,n\}$.

Properties: $\left\{ {n \atop 1} \right\}=1$, and $\left\{ {n \atop n} \right\}=1$ with $\left\{ {n \atop 0} \right\}=\delta_{n0}$.

For $n=4$:

$k=1$	$k=2$	$k=3$	$k=4$
$\{\{1,2,3,4\}\}$	$\{\{1\},\{2,3,4\}\}$		$\{\{1\},\{2\},\{3\},\{4\}\}$
	$\{\{2\},\{1,3,4\}\}$	$\{\{1\},\{2\},\{3,4\}\}$	
	$\{\{3\},\{1,2,4\}\}$	$\{\{1\},\{3\},\{2,4\}\}$	
	$\{\{4\},\{1,2,3\}\}$	$\{\{1\},\{4\},\{2,3\}\}$	
	$\{\{1,2\},\{3,4\}\}$	$\{\{2\},\{3\},\{1,4\}\}$	
	$\{\{1,3\},\{2,4\}\}$	$\{\{2\},\{4\},\{1,3\}\}$	
	$\{\{1,4\},\{2,3\}\}$	$\{\{3\},\{4\},\{1,2\}\}$	

For example, Find the value of $\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\}$.

The set $\{1,2,3,4\}$ can split into two non-empty subsets in the following ways:

$\{1,2,3\}\cup\{4\}$, $\{1,2,4\}\cup\{3\}$, $\{1,3,4\}\cup\{2\}$, $\{2,3,4\}\cup\{1\}$,

$\{1,2\}\cup\{3,4\}$, $\{1,3\}\cup\{2,4\}$, and $\{1,4\}\cup\{2,3\}$.

On other word, placing the 4 distinguishable balls $\{a,b,c,d\}$ into 2 indistinguishable cells, so that no cell empty, can be denote in 7 ways. These are (vertical bars delineate the cells):

$|ab|cd|$, $|ad|bc|$, $|ac|bd|$, $|a|bcd|$, $|b|acd|$, $|c|abd|$, and $|a|bcd|$.

Thus, $\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} = 7$.

Lemma (1) (Recurrence relation)

Stirling numbers of the second kind satisfy the recurrence relation

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} \quad \text{for} \quad 0 < k \le n. \qquad \text{(I)}$$

Proof. To prove recurrence (I) observe that the set partitions of $[n]$ with exactly k subsets fall into one of two categories; those which contain the subset $\{n\}$ and those which do not:

- If the set partition contains $\{n\}$ (i.e., $\{n\}$ is one of the subsets), delete this subset and obtain a one-to- one correspondence with a set partition of $[n-1]$ consisting of $k-1$ parts. That is there are $\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$ ways of decomposing $[n] \setminus \{n\}$ into $k-1$ non-empty subsets.
- If $\{n\}$ is not a subset of the set partition of $[n]$, this set partition came by inserting n into one of the k subsets of a set partition of $[n-1]$. That is we decompose $[n] \setminus \{n\}$ into k non-empty subsets in $\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$ ways. We can place n in any of these k subsets. So, there are $\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$ ways of doing this.

To understand the recurrence (I), observe that a partition of the n objects into k nonempty subsets either contains the subset $\{n\}$ or it does not. The number of ways that $\{n\}$ is one of the subsets is given by $\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$, since we must partition the remaining $n-1$ objects into the available $k-1$ subsets. The number of ways that $\{n\}$ is not one of the subsets (that is, n belongs to a subset containing other elements) is given by $k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}$, since we partition all elements other than n into k subsets, and then are left with k choices for inserting the element n. Summing these two values gives the desired result.

__**Lemma (2)**. (Closed form expression)__

Closed form expression for $S(n,\alpha) \equiv \begin{Bmatrix} n \\ \alpha \end{Bmatrix}$, is given by

$$\begin{Bmatrix} n \\ \alpha \end{Bmatrix} = \sum_{k=1}^{\alpha} (\alpha - k + 1) \begin{Bmatrix} n-k \\ \alpha - k + 1 \end{Bmatrix}, \text{ for } \alpha \geq 1 \text{ and, } n \neq \alpha . \qquad \text{(II)}$$

Proof. Multiply equation (I), by $k!$, then let $k \to \alpha$, and $n \to n-1$.

$$\alpha! \begin{Bmatrix} n \\ \alpha \end{Bmatrix} = \alpha \, \alpha! \begin{Bmatrix} n-1 \\ \alpha \end{Bmatrix} + \alpha (\alpha - 1)! \begin{Bmatrix} n-1 \\ \alpha - 1 \end{Bmatrix}. \qquad (1)$$

Take equation (1) and let $n \to n-1$ and $\alpha \to \alpha - 1$ to obtain

$$(\alpha-1)! \begin{Bmatrix} n-1 \\ \alpha-1 \end{Bmatrix} = (\alpha-1)(\alpha-1)! \begin{Bmatrix} n-2 \\ \alpha-1 \end{Bmatrix} + (\alpha-1)(\alpha-2)! \begin{Bmatrix} n-2 \\ \alpha-2 \end{Bmatrix}. \qquad (2)$$

Substitute Equation (2) into (1). This gives us

$$(\alpha)! \begin{Bmatrix} n \\ \alpha \end{Bmatrix} = \alpha (\alpha)! \begin{Bmatrix} n-1 \\ \alpha \end{Bmatrix} + \alpha(\alpha-1)(\alpha-1)! \begin{Bmatrix} n-2 \\ \alpha-1 \end{Bmatrix}$$

$$+\alpha(\alpha-1)(\alpha-2)! \begin{Bmatrix} n-2 \\ \alpha-2 \end{Bmatrix}. \qquad (3)$$

Now repeat this process. Take equation (1), let $n \to n-2$ and

$\alpha \to \alpha - 2$ to obtain

$$(\alpha-2)!\left\{ \begin{matrix} n-2 \\ \alpha-2 \end{matrix} \right\} = (\alpha-2)(\alpha-2)!\left\{ \begin{matrix} n-3 \\ \alpha-2 \end{matrix} \right\} + (\alpha-2)(\alpha-3)!\left\{ \begin{matrix} n-2 \\ \alpha-2 \end{matrix} \right\}. \quad (4)$$

Substitute (4) into Equation (3). This gives us

$$\alpha!\left\{ \begin{matrix} n \\ \alpha \end{matrix} \right\} = \alpha\alpha!\left\{ \begin{matrix} n-1 \\ \alpha \end{matrix} \right\} + \alpha(\alpha-1)(\alpha-1)!\left\{ \begin{matrix} n-2 \\ \alpha-1 \end{matrix} \right\}$$

$$+\alpha(\alpha-1)(\alpha-2)(\alpha-2)!\left\{ \begin{matrix} n-3 \\ \alpha-2 \end{matrix} \right\} + \alpha(\alpha-1)(\alpha-2)(\alpha-3)!\left\{ \begin{matrix} n-3 \\ \alpha-3 \end{matrix} \right\}. \quad (5)$$

After m iterations we find

$$\alpha!\left\{ \begin{matrix} n \\ \alpha \end{matrix} \right\} = \sum_{k=0}^{m} \binom{\alpha}{k+1}(k+1)!(\alpha-k)!\left\{ \begin{matrix} n-1-k \\ \alpha-k \end{matrix} \right\}$$

$$+\binom{\alpha}{m+1}(m+1)!(\alpha-m-1)!\left\{ \begin{matrix} n-m-1 \\ \alpha-m-1 \end{matrix} \right\}. \qquad (6)$$

In equation (6) let $m = \alpha - 1$ to obtain

$$\alpha!\left\{ \begin{matrix} n \\ \alpha \end{matrix} \right\} = \sum_{k=0}^{\alpha-1} \binom{\alpha}{k+1}(k+1)!(\alpha-k)!\left\{ \begin{matrix} n-1-k \\ \alpha-k \end{matrix} \right\} + \alpha!\left\{ \begin{matrix} n-m-1 \\ 0 \end{matrix} \right\}. \qquad (7)$$

As long as $n - m - 1 \neq 0$, $\left\{ \begin{matrix} n-m-1 \\ 0 \end{matrix} \right\} = 0$, and equation (7) becomes

$$\alpha!\left\{ \begin{matrix} n \\ \alpha \end{matrix} \right\} = \sum_{k=0}^{\alpha-1} \binom{\alpha}{k+1}(k+1)!(\alpha-k)!\left\{ \begin{matrix} n-1-k \\ \alpha-k \end{matrix} \right\}, \text{ or}$$

$$\alpha!\left\{ \begin{matrix} n \\ \alpha \end{matrix} \right\} = \sum_{k=1}^{\alpha} \binom{\alpha}{k} k!(\alpha-k+1)!\left\{ \begin{matrix} n-k \\ \alpha-k+1 \end{matrix} \right\}. \qquad (8)$$

Take Equation (8), divided by α, and simplify to obtain

$$\left\{ \begin{matrix} n \\ \alpha \end{matrix} \right\} = \sum_{k=1}^{\alpha} (\alpha - k + 1) \left\{ \begin{matrix} n-k \\ \alpha - k + 1 \end{matrix} \right\}, \text{ for } \alpha \geq 1 \text{ and, } n \neq \alpha . \qquad (9)$$

Equation (9), when combined with closed forms for $\sum_{k=0}^{n-m} k^r$, recursively provides polynomials for $\left\{ \begin{matrix} n \\ n-m \end{matrix} \right\}$ whenever m is a positive integer.

Facts:

1- Calculate the value of $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}$.

Solution. Use equation (9) with $\alpha = n-1$, we observe that

$$\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \sum_{k=1}^{n-1} (n-k) \left\{ \begin{matrix} n-k \\ n-k \end{matrix} \right\} = \sum_{k=1}^{n-1} (n-k) = n(n-1) - \frac{n(n-1)}{2} = \binom{n}{2}.$$

2- Calculate the value of $\left\{ \begin{matrix} n \\ n-2 \end{matrix} \right\}$.

Solution. Use equation (9) with $\alpha = n-2$, along with $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2}$:

$$\left\{ \begin{matrix} n \\ n-2 \end{matrix} \right\} = \sum_{k=1}^{n-2} (n-k+1) \left\{ \begin{matrix} n-k \\ n-k+1 \end{matrix} \right\} = \sum_{k=1}^{n-2} (n-k+1) \binom{n-k}{2}$$

$$= \sum_{k=1}^{n-2} \frac{(n-k+1)^2 (n-k)}{2} = \frac{1}{2} \sum_{k=1}^{n-2} k^2 (k+1) = \frac{1}{2} \left[\sum_{k=1}^{n-2} k^3 + \sum_{k=1}^{n-2} k^2 \right]$$

$$= \frac{1}{2}\left[\frac{(n-2)^2(n-1)^2}{4} + \frac{(n-2)(n-1)(2n-3)}{6}\right]$$

$$= \frac{n(n-1)(n-2)(3n-5)}{24}.$$

Continuing this recursive procedure, we can show that

$$\left\{ \begin{matrix} n \\ n-3 \end{matrix} \right\} = \frac{n(n-1)(n-2)^2(n-3)^2}{48},$$

$$\left\{ \begin{matrix} n \\ n-4 \end{matrix} \right\} = \frac{n(n-1)(n-2)(n-3)(n-4)(15n^3-150n^2+485n-502)}{5760}, \text{ and}$$

$$\left\{ \begin{matrix} n \\ n-5 \end{matrix} \right\} = \frac{n(n-1)(n-2)(n-3)(n-4)^2(n-5)^2(3n^2-23n+38)}{5760}.$$

<u>Example</u>. Find the value of $\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\}$.

Solution. Using the above fact, we obtain $\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} = \binom{4}{2} = \frac{4\times 3}{2} = 6$.

Example. Find the value of $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\}$.

Solution. Using the above fact, we obtain

from (I), we get $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} = \binom{4}{2} + 3\binom{4}{3} = 7 + 3\times 6 = 25$.

From, $\left\{ \begin{matrix} n \\ n-2 \end{matrix} \right\} = \frac{n(n-1)(n-2)(3n-5)}{24}$, by putting $n=5$, we have

$$\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} = \frac{5\times 4\times 3\times(3\times 5-5)}{24} = \frac{600}{24} = 25.$$

Lemma (3). Closed form expression for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, is given by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left(\sum_{m=0}^{L-1} (k-m) \left\{ \begin{matrix} n-m-1 \\ k-m \end{matrix} \right\} \right) + \left\{ \begin{matrix} n-L \\ k-L \end{matrix} \right\}.$$

Proof. Using the recursion $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$ we get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + (k-1) \left\{ \begin{matrix} n-2 \\ k-1 \end{matrix} \right\} + \left\{ \begin{matrix} n-2 \\ k-2 \end{matrix} \right\}$$

$$= \left(\sum_{m=0}^{1} (k-m) \left\{ \begin{matrix} n-m-1 \\ k-m \end{matrix} \right\} \right) + \left\{ \begin{matrix} n-2 \\ k-2 \end{matrix} \right\} = \ldots$$

$$= \left(\sum_{m=0}^{L-1} (k-m) \left\{ \begin{matrix} n-m-1 \\ k-m \end{matrix} \right\} \right) + \left\{ \begin{matrix} n-L \\ k-L \end{matrix} \right\}.$$

The following Lemma gives the basis definition of $\left\{ \begin{matrix} n \\ j \end{matrix} \right\} \equiv S(n, j)$.

Lemma (4). Recurrence (8) implies the basis definition of Stirling numbers of the second kind, $\left\{ \begin{matrix} n \\ j \end{matrix} \right\} \equiv S(n, j)$:

$$x^n = \sum_{j=0}^{n} \binom{x}{j} j! S(n, j) \equiv \sum_{j=0}^{n} \binom{x}{j} j! \left\{ \begin{matrix} n \\ j \end{matrix} \right\}, \qquad (10)$$

whenever n is a nonnegative integer.

Proof. Let $\{C(n, j)\}_{j=0}^{n}$ be the coefficient of $\binom{x}{j}$ in the expansion

$$x^n = \sum_{j=0}^{n} \binom{x}{j} j! C(n, j).$$

Notice that $C(0,0)=1$ and $C(n,j)=0$ if $j>n$.

By definition we have, $x^{n+1}=\sum_{j=0}^{n+1}\binom{x}{j}j!C(n+1,j)$. On the other hand

$$\begin{aligned}
x^{n+1}=x.x^{n}&=x\sum_{j=0}^{n}\binom{x}{j}j!C(n,j)\\
&=\sum_{j=0}^{n}(x-j)\binom{x}{j}j!C(n,j)+\sum_{j=0}^{n}j\binom{x}{j}j!C(n,j)\\
&=\sum_{j=0}^{n+1}(x-j+1)\binom{x}{j-1}(j-1)!C(n,j-1)+\sum_{j=0}^{n}j\binom{x}{j}j!C(n,j)\\
&=\sum_{j=0}^{n+1}\binom{x}{j}j!C(n,j-1)+\sum_{j=0}^{n+1}j\binom{x}{j}j!C(n,j)\\
&=\sum_{j=0}^{n+1}\left[C(n,j-1)+jC(n,j)\right]j!\binom{x}{j}.
\end{aligned}$$

Comparing the coefficient of $j!\binom{x}{j}$ implies that

$$C(n+1,j)=jC(n,j)+C(n,j-1).$$

This is precisely the recurrence obeyed by $\left\{ n \atop j \right\}$.

Since $C(0,0)=\left\{ 0 \atop 0 \right\}=1$ and $C(n,j)=0=\left\{ n \atop j \right\}$ if $j>n$, we conclude that $C(n,j)=\left\{ n \atop j \right\}$ for all nonnegative integers j and n.

Lemma (3) provides easy proofs of the following three identities:

$$\sum_{j=0}^{n}(-1)^j\, j!\left\{ n \atop j \right\} = (-1)^n , \tag{11}$$

$$\sum_{j=0}^{n}(-1)^j \binom{x+j-1}{j} j!\left\{ n \atop j \right\} = (-1)^n x^n , \tag{12}$$

$$\sum_{j=0}^{n}(-1)^j \binom{2j}{j} \frac{j!}{2^{2j}}\left\{ n \atop j \right\} = \frac{(-1)^n}{2^n} . \tag{13}$$

Equation (13) is equation (10) with $x=-1/2$; equation (11) is equation (10) with $x=-1$ while equation (12) is equation (10) with $x \to -x$. If $x=1$, equation (12) becomes equation (11).

This suggests that we take equation (12) and let $x=2$ to find that

$$\sum_{j=0}^{n}(-1)^j (j+1)\, j!\left\{ n \atop j \right\} = \sum_{j=0}^{n}(-1)^j\, jj!\left\{ n \atop j \right\} + \sum_{j=0}^{n}(-1)^j\, j!\left\{ n \atop j \right\} = (-1)^n 2^n . \tag{14}$$

Use Equation (11) to evaluate the right most sum of Equation (14). After simplification we obtain the identity

$$\sum_{j=0}^{n}(-1)^j\, jj!\left\{ n \atop j \right\} = (-1)^n \left[2^n - 1 \right] . \tag{15}$$

Appendices:

Preliminaries: Sample Selection

There are 4 different ways in which a sample of k elements can be obtained from a set of n distinguishable objects.

Order counts	Repetitions allowed	The sample is called an	Number of ways to choose the sample
No	No	k-combination	$\binom{n}{k}$
Yes	No	k-permutation	$(n)_k = \frac{n!}{(n-k)!}$
No	Yes	k-combination with replacement	$\binom{n+k-1}{k}$
Yes	Yes	k-permutation with replacement	n^k

Balls into cells

There are 8 different ways in which n balls can be placed into k cells:

Distinguish the balls?	Distinguish the cells?	Can cells be empty?	Number of ways to place n balls into k cells
Yes	Yes	Yes	k^n
Yes	Yes	No	$k!\left\{ {n \atop k} \right\}$
Yes	No	Yes	$\left\{ {n \atop 1} \right\} + \left\{ {n \atop 2} \right\} + \ldots + \left\{ {n \atop k} \right\}$
Yes	No	No	$\left\{ {n \atop k} \right\}$
No	Yes	Yes	

No	Yes	No	$\binom{k+n-1}{n}$
No	No	Yes	$\binom{n-1}{k-1}$
No	No	No	$p_1(n)+p_2(n)+...+p_k(n)$
			$p_k(n)$

where

$\left\{ n \atop k \right\}$: is the Stirling number of the second kind,

$p_k(n)$: is the number of partitions of the number n into exactly k integer parts.

Relations.

Let x be any real number and consider the function $f(x)=(x)_x$, we have

$$f(x)=(x)_x = x(x)_{x-1} = xf(x-1).$$

But the gamma function $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ has $\Gamma(x+1)=x\Gamma(x)$, as functional equation. Hence when x is not negative integer we define $(x)_x = \Gamma(x+1), \quad x \neq -1,-2,...$, then we have $(x)_x = (x)_{n+x-n} = (x)_n (x-n)_{x-n}$, and the general definitions of the factorial power is $(x)_n := \dfrac{\Gamma(x+1)}{\Gamma(x+1-n)}$, for all real values of x and n for which the gamma functions exist.