Chapter 3

The Lebesgue integral

3.1 Introduction

We now turn our attention to the construction of the Lebesgue integral of general functions, which, as already discussed, is necessary to avoid the technical deficiencies associated with the Riemann integral.

3.2 Measurable functions

3.2.1 Basic notions

The **extended real number system**, $\overline{\mathbb{R}}$, is the set of real numbers together with two symbols $-\infty$ and $+\infty$. That is, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, \infty]$. The algebraic operations for these two infinities are:

[1] For $r \in \mathbb{R}$, $\pm\infty + r = \pm\infty$.

[2] For $r \in \mathbb{R}$, $r(\pm\infty) = \pm\infty$, if $r > 0$ and $r(\pm\infty) = \mp\infty$ if $r < 0$.

[3] $+\infty + (+\infty) = +\infty$ and $-\infty + (-\infty) = -\infty$.

[4] $\infty + (-\infty)$ is undefined.

[5] $0 \cdot (\pm\infty) = 0$.

3.2.1 Definition
Let (X, Σ) be a measurable space and $E \in \Sigma$. A function $f : E \to \overline{\mathbb{R}}$ is said to be **measurable** if for each $\alpha \in \mathbb{R}$, the set $\{x \in E : f(x) > \alpha\}$ is measurable.

3.2.2 Proposition
Let (X, Σ) be a measurable space and $E \in \Sigma$, and $f \to \overline{R}$. Then the following are equivalent:

[1] f is measurable.

[2] For each $\alpha \in \mathbb{R}$, the set $\{x \in E : f(x) \geq \alpha\}$ is measurable.

[3] For each $\alpha \in \mathbb{R}$, the set $\{x \in E : f(x) < \alpha\}$ is measurable.

[4] For each $\alpha \in \mathbb{R}$, the set $\{x \in E : f(x) \leq \alpha\}$ is measurable.

PROOF.

[1] $\Rightarrow$ [2]: If f is measurable, then the set

$$\{x \in E : f(x) \geq \alpha\} = \bigcap_{n \in \mathbb{N}} \left\{x \in E : f(x) > \alpha - \frac{1}{n}\right\},$$

which is an intersection of measurable sets, is measurable.

[2] $\Rightarrow$ [3]: If the set $\{x \in E : f(x) \geq \alpha\}$ is measurable, then so is the set

$$E \backslash \{x \in E : f(x) \geq \alpha\} = \{x \in E : f(x) < \alpha\}.$$

[3] $\Rightarrow$ [4]: If the set $\{x \in E : f(x) < \alpha\}$ is measurable, then so is the set

$$\{x \in E : f(x) \leq \alpha\} = \bigcap_{n \in \mathbb{N}} \left\{x \in E : f(x) < \alpha + \frac{1}{n}\right\}$$

since it is an intersection of measurable sets.

[4] $\Rightarrow$ [1]: If the set $\{x \in E : f(x) \leq \alpha\}$, is measurable, so is its complement in E. Hence,

$$\{x \in E : f(x) > \alpha\} = E \backslash \{x \in E : f(x) \leq \alpha\}$$

is measurable. It follows from this that f is measurable.

$\square$

3.2.3 Examples

[1] The constant function is measurable. That is, if (X, Σ) is a measurable space, $c \in \mathbb{R}$, and $E \in \Sigma$, then the function $f : E \to \mathbb{R}$ given by $f(x) = c$, for each $x \in E$, is measurable.

Let $\alpha \in \mathbb{R}$. If $\alpha \geq c$, then the set $\{x \in E : f(x) > \alpha\} = \emptyset$, and is thus measurable.

If $\alpha < c$, then the set $\{x \in E : f(x) > \alpha\} = E$, and is therefore measurable. Hence, f is measurable.

[2] Let (X, Σ) be a measurable space and let $A \in \Sigma$. The characteristic function, χ_A, is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The characteristic function χ_A is measurable. This follows immediately from the observation that, for each $\alpha \in \mathbb{R}$ and $E \in \Sigma$, the set $\{x \in E : \chi_A > \alpha\}$ is either E, A or $\emptyset$.

[3] Let $\mathbb{B}$ be the Borel σ-algebra in $\mathbb{R}$ and $E \in \mathbb{B}$. Then, any continuous function $f : E \to \overline{\mathbb{R}}$ is measurable. This is an immediate consequence of the fact that, if $f : E \to \overline{\mathbb{R}}$ is continuous and $\alpha \in \mathbb{R}$, then the set $\{x \in E : f(x) > \alpha\}$ is open and hence belongs to $\mathbb{B}$.

3.2.4 Proposition

[1] If f and g are measurable real-valued functions defined on a common domain $E \in \Sigma$ and $c \in \mathbb{R}$, then the functions

(a) $f + c$,

(b) cf,

(c) $f \pm g$,

(d) f^2,

(e) $f \cdot g$,

(f) $|f|$,

(g) $f \vee g$,

(h) $f \wedge g$ are also measurable.

[2] If (f_n) is a sequence of measurable functions defined on a common domain $E \in \Sigma$, then the functions

(a) $\sup_n f_n$,

(b) $\inf_n f_n$,

(c) $\limsup_n f_n$,

(d) $\liminf_n f_n$ are also measurable.

PROOF.

[1] (a) For any real number α,

$$\{x \in E : f(x) + c > \alpha\} - \{x \in E : f(x) > \alpha - c\}.$$

Since the set on the right-hand side is measurable, we have that $f + c$ is measurable.

(b) If $c = 0$, then cf is obviously measurable. Assume that $c < 0$. Then, for each real number α,

$$\{x \in E : cf(x) > \alpha\} = \left\{x \in E : f(x) < \frac{\alpha}{c}\right\}.$$

Since the set $\{x \in E : f(x) < \frac{\alpha}{c}\}$ is measurable, it follows that cf is also measurable.

(c) Let α be a real number. Since the rationals are dense in the reals, there is a rational number r such that

$$f(x) < r < \alpha - g(x),$$

whenever $f(x) + g(x) < \alpha$. Therefore,

$$\{x \in E : f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x \in E : f(x) < r\} \cap \{x \in E : g(x) < \alpha - r\}).$$

Since the sets $\{x \in E : f(x) < r\}$ and $\{x \in E : g(x) < \alpha - r\}$ are measurable, so is the set $\{x \in E : f(x) < r\} \cap \{x : g(x) < \alpha - r\}$, and consequently, the set $\{x \in E : f(x) + g(x) < \alpha\}$, being a countable union of measurable sets, is also measurable.

If g is measurable, it follows from (b) that $(-1)g$ is also measurable. Hence, so is $f + (-1)g = f - g$.

(d) Let $\alpha \in \mathbb{R}$ and $E \in \Sigma$. If $\alpha < 0$, then $\{x \in E : f^2(x) > \alpha\} = E$, which is measurable.

If $\alpha \geq 0$, then

$$\{x \in E : f^2(x) > \alpha\} = \{x \in E : f(x) > \sqrt{\alpha}\} \cup \{x \in E : f(x) < -\sqrt{\alpha}\}.$$

Since the two sets on the right hand side are measurable, it follows that the set $\{x \in E : f^2(x) > \alpha\}$ is also measurable. Hence f^2 is measurable.

(e) Since $f \cdot g = \frac{1}{4}[(f+g)^2 - (f-g)^2]$, it follows from (b), (c), and (d) that $f \cdot g$ is measurable.

(f) Let $\alpha \in \mathbb{R}$ and $E \in \Sigma$. If $\alpha < 0$, then $\{x \in E : |f(x)| > \alpha\} = E$, which is measurable.

If $\alpha \geq 0$, then

$$\{x \in E : |f(x)| > \alpha\} = \{x \in E : f(x) > \alpha\} \cup \{x \in E : f(x) < -\alpha\}.$$

Since the two sets on the right hand side are measurable, it follows that the set $\{x \in E : f(x)^2 > \alpha\}$ is also measurable. Thus, $|f|$ is measurable.

(g) It is sufficient to observe that $f \vee g = \frac{1}{2}\{f + g + |f - g|\}$. It now follows from (b), (c), and (d) that $f \vee g$ is measurable.

(h) It is sufficient to observe that $f \wedge g = \frac{1}{2}\{f + g - |f - g|\}$. It now follows from (b), (c), and (d) that $f \wedge g$ is measurable.

[2] (a) Let $\alpha \in \mathbb{R}$. Then

$$\{x \in E : \sup_n f_n > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) > \alpha\}.$$

Since for each $n \in \mathbb{N}$, f_n is measurable, it follows that the set $\{x \in E : f_n(x) > \alpha\}$ is measurable for each $n \in \mathbb{N}$. Therefore, the set $\{x \in E : \sup_n f_n(x) > \alpha\}$ is measurable as it is a countable union of measurable sets.

(b) It is sufficient to note that $\inf_n f_n = -\sup_n(-f_n)$.

(c) Notice that $\limsup_n f_n = \inf_{n\geq 1}\left(\sup_{k\geq n} f_k\right)$ and use (a) and (b) above.

(d) Notice that $\liminf_n f_n = \sup_{n\geq 1}\left(\inf_{k\geq n} f_k\right)$ and use (a) and (b) above.

□

3.2.5 Corollary
Let (X, Σ) be a measurable space. If f is a pointwise limit of a sequence (f_n) of measurable functions defined on a common domain $E \in \Sigma$, then f is measurable.

PROOF.

If the sequence (f_n) converges pointwise to f, then, for each $x \in E$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_n f_n(x).$$

Now, by Proposition 3.2.4 [2], f is measurable.

□

3.2.6 Definition
Let f be a real-valued function defined on a set X. The **positive part of** f, denoted by f^+, is the function $f^+ = \max\{f, 0\} = f \vee 0$ and the **negative part of** f, denoted by f^-, is the function $f^- = \max\{-f, 0\} = (-f) \vee 0$.

Immediately we have that if f is a real-valued function defined on X, then

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$

Note that

$$f^+ = \frac{1}{2}[|f| + f] \quad \text{and} \quad f^- = \frac{1}{2}[|f| - f].$$

Let (X, Σ) be a measurable space and $f : X \to \mathbb{R}$. It is trivial to deduce from Proposition 3.2.4 that f is measurable if and only if f^+ and f^- are measurable.

3.2.7 Definition
Let (X, Σ) be a measurable space. A **simple function** on X is a function of the form $\phi = \sum_{j=1}^n c_j \chi_{E_j}$, where, for each $j = 1, 2, \ldots n$, c_j is an extended real number and $E_j \in \Sigma$.

Since χ_{E_j} is measurable for each $j = 1, 2, \ldots, n$, it follows from Proposition 3.2.4 that ϕ is also measurable.

[2] $f \leq g$ a.e. if $\mu(\{x \in X : f(x) > g(x)\}) = 0$.

3.2.12 Definition
Let (X, Σ, μ) be a measure space. A function $f : X \to \overline{\mathbb{R}}$ is said to be **almost everywhere real-valued**, denoted by a.e. real-valued, if

$$\mu(\{x \in X : |f(x)| = \infty\}) = 0.$$

We call a set of measure zero a **null set**.

3.2.13 Definition
A measure space (X, Σ, μ) is said to be **complete** if Σ contains all subsets of sets of measure zero. That is, if $E \in \Sigma$, $\mu(E) = 0$ and $A \subset E$, then $A \in \Sigma$.

It follows from Theorem 2.3.6 [1] that if a measure space (X, Σ, μ) is complete, $E \in \Sigma$, $\mu(E) = 0$ and $A \subset E$, then $\mu(A) = 0$.

3.2.14 Proposition
Let (X, Σ, μ) be a complete measure space and $f = g$ a.e. If f is measurable on $E \in \Sigma$, then so is g.

PROOF.

Let $\alpha \in \mathbb{R}$ and $N = \{x \in E : g(x) \neq f(x)\}$. Then $N \in \Sigma$ and $\mu(N) = 0$. Now,

$$\begin{aligned}\{x \in E : g(x) > \alpha\} &= \{x \in E \backslash N : g(x) > \alpha\} \cup \{x \in N : g(x) > \alpha\} \\ &= \{x \in E \backslash N : f(x) > \alpha\} \cup \{x \in N : g(x) > \alpha\}.\end{aligned}$$

The first set on the right hand side is measurable since f is measurable. The second set is measurable since it is a subset of the null set N and the measure is complete.

□

3.2.2 Convergence of sequences of measurable functions

We now consider various notions of convergence of a sequence of measurable functions examine the relationships that exist between them.

3.2.15 Definition
Let $(X \Sigma, \mu)$ be a measure space. A sequence (f_n) of a.e. real-valued measurable functions on X is said to

[1] **converge almost everywhere** to an a.e. real-valued measurable function f. denoted by $f_n \to^{\text{a.e.}} f$, if for each $\epsilon > 0$ and each $x \in X$, there is a set $E \in \Sigma$ and a natural number $N = N(\epsilon)$ such that $\mu(E) < \epsilon$ and

$$|f_n(x) - f(x)| < \epsilon, \text{ for each } x \in X \backslash E \text{ and each } n \geq N.$$

[2] **converge almost uniformly** to an a.e. real-valued measurable function f, denoted by $f_n \to^{\text{a.u.}} f$, if for each $\epsilon > 0$, there is a set $E \in \Sigma$ and a natural number $N = N(\epsilon)$ such that $\mu(E) < \epsilon$ and

$$\|f_n - f\|_\infty = \sup_{x \in X \backslash E} |f_n(x) - f(x)| < \epsilon, \text{ for each } n \geq N.$$

[3] **converge in measure** to an a.e. real-valued measurable function f, denoted by $f_n \to^{\mu} f$, if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

(if μ is a probability measure, then this mode of convergence is called **convergence in probability**).

3.2.16 Proposition
Let (X, Σ, μ) be a complete measure space and (f_n) a sequence of measurable functions on $E \in \Sigma$ which converges to f a.e. Then f is measurable on E.

PROOF.

Let $\alpha \in \mathbb{R}$ and $N = \{x \in E : \lim_{n\to\infty} f_n(x) \neq f(x)\}$. Then $N \in \Sigma, \mu(N) = 0$ and $\lim_{n\to\infty} f_n(x) = f(x)$, for each $x \in E\backslash N$. Since E and N are measurable, so is $E\backslash N$. For each $n \in \mathbb{N}$, define g_n and g by

$$g_n(x) = \begin{cases} 0 & \text{if } x \in N \\ f_n(x) & \text{if } x \in E\backslash N. \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if } x \in N \\ f(x) & \text{if } x \in E\backslash N. \end{cases}$$

Then $g_n = f_n$ a.e. and $f = g$ a.e. It follows from Proposition 3.2.4, that for each $n \in \mathbb{N}$, g_n is measurable. If $x \in N$, then

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} 0 = 0 = g(x), \tag{3.1}$$

and if $x \in E\backslash N$, then

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n = 0 = f(x) = g(x) \tag{3.2}$$

It follows from (3.1) and (3.2) that the sequence (g_n) converges pointwise to g *everywhere* on E. By Corollary 3.2.5, that g is measurable. Since $g = f$ a.e., we have, by Proposition 3.2.14, that f is measurable.

□

3.2.17 Theorem

Let (X, Σ, μ) be a measure space and (f_n) be a sequence of real-valued measurable functions on X. If the sequence (f_n) converges almost uniformly to f, then it converges in the measure to f. That is, almost uniform convergence implies convergence in the measure.

PROOF.

Let $\epsilon > 0$ be given. Since the sequence (f_n) converges almost uniformly to f, there is a measurable set E and a natural number N such that $\mu(E) < \epsilon$ and

$$|f_n(x) - f(x)| < \epsilon, \text{ for all } x \in X\backslash E \text{ and all } n \geq N.$$

It now follows that, for all $n \geq N$, $\{x \in X : |f_n(x) - f(x)| \geq \epsilon\} \subset E$. Therefore, for all $n \in \mathbb{N}$,

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon.$$

□

3.2.18 Theorem

Let (X, Σ, μ) be a measure space and (f_n) be a sequence of a.e. real-valued measurable functions on X. If the sequence (f_n) converges almost uniformly to f, then it converges almost everywhere to f. That is, almost uniform convergence implies convergence almost everywhere.

PROOF.

Suppose that the sequence (f_n) converges almost uniformly to f. Then, for each $n \in \mathbb{N}$, there is a measurable set E_n, with $\mu(E_n) < \frac{1}{n}$ such that $f_n \to f$ uniformly on $X\backslash E_n$. Let $A = \bigcup_{n=1}^{\infty}(X\backslash E_n)$. Then

$$\mu(X\backslash A) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \leq \mu(E_n) = \frac{1}{n} \to 0.$$

3.3 Definition of the integral

In this section, we define a Lebesgue integral. This is done in three stages: firstly, we define the integral of a nonnegative simple function; then, using the integral of a nonnegative simple function, we define the integral or any measurable function. We also prove three key results: Monotone Convergence Theorem, Dominated Convergence Theorem and Fatou's Lemma.

Unless otherwise specified, we will work in the measure space (X, Σ, μ).

3.3.1 Integral of a nonnegative simple function

3.3.1 **Definition**
Let ϕ be a nonnegative simple function with the canonical representation $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$. The **Lebesgue integral of ϕ with respect to** μ, denoted by $\int_X \phi d\mu$, is the extended real number

$$\int_X \phi d\mu = \sum_{i=1}^{n} a_i \mu(E_i).$$

If $A \in \Sigma$, we define

$$\int_A \phi d\mu = \int_X \chi_A \phi d\mu.$$

The function ϕ is integrable if $\int_X \phi d\mu$ is finite.

We also write $\int \phi d\mu$ for $\int_X \phi d\mu$.

It is straightforward to show that if A is a measurable set and ϕ is as above, then

$$\int_A \phi d\mu = \sum_{i=1}^{n} a_i \mu(A \cap E_i).$$

Furthermore,

$$\int_A d\mu = \int_A \chi_A d\mu = \int_X \chi_A d\mu = \mu(A).$$

It is necessary to show that the above definition of the Lebesgue integral is unambiguous, i.e, the integral is independent of the representation of ϕ. Assume that

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i} = \sum_{j=1}^{m} b_j \chi_{F_i},$$

where $E_i \cap E_k = \emptyset$, for all $1 \le i \ne k \le n$, $X = \bigcup_{i=1}^{n} E_i$; $F_k \cap F_l = \emptyset$, for all $1 \le j \ne l \le m$, and $\bigcup_{j=1}^{m} F_j$. Then for each $i = 1, 2, \dots, n$ and each $j = 1, 2, \dots, m$,

$$E_i = E_i \cap X = E_i \cap \left(\bigcup_{j=1}^{m} F_j \right) = \bigcup_{j=1}^{m} (E_i \cap F_j), \text{ a disjoint union, and}$$

$$F_j = F_j \cap X = \left(\bigcup_{i=1}^{n} E_i \right) = \bigcup_{i=1}^{n} (E_i \cap F_j), \text{ a disjoint union.}$$

If $E_i \cap F_j \neq \emptyset$, then, for $x \in E_i \cap F_j$, $a_i = \phi(x) = b_j$. That is, $a_i = b_j$ in this case. If $E_i \cap F_j = \emptyset$, then $\mu(E_i \cap F_j) = 0$. Thus, $a_i \mu(E_i \cap F_j) = b_j \mu(E_i \cap F_j)$, for all $1 \le i \le n$ and for all $1 \le j \le m$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} a_i \mu(E_i) &= \sum_{i=1}^{n} a_i \mu\Big(\bigcup_{j=1}^{m} (E_i \cap F_j)\Big) = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(E_i \cap F_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \mu(E_i \cap F_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j \mu(E_i \cap F_j) \\ &= \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mu(E_i \cap F_j) = \sum_{j=1}^{m} b_j \mu\Big(\bigcup_{i=1}^{n} (E_i \cap F_j)\Big)\end{aligned}$$

$$\sum_{j=1}^{m} b_j \mu(F_j).$$

Hence the definition of the integral of ϕ is unambiguous.

3.3.2 Proposition

Let (X, Σ, μ) be a measure space, ϕ and ψ nonnegative simple functions, and c a nonnegative real number. Then

[1] $\int_X (\phi + \psi) d\mu = \int_X \phi d\mu + \int_X \psi d\mu$.

[2] $\int_X c\phi d\mu = c \int_X \phi d\mu$.

[3] If $\phi \le \psi$, then $\int_X \phi d\mu \le \int_X \psi d\mu$.

[4] If A and B are disjoint measurable sets, then

$$\int_{A\cup B} \phi d\mu = \int_A \phi d\mu + \int_B \phi d\mu.$$

[5] The set function $\nu : \Sigma \to [0, \infty]$ defined by

$$\nu(A) = \int_A \phi d\mu,$$

for $A \in \Sigma$, is a measure on X.

[6] If $A \in \Sigma$ and $\mu(A) = 0$, then $\int_A \phi d\mu = 0$.

PROOF.

Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, and $\psi = \sum_{j=1}^{m} b_j \chi_{F_j}$ be canonical representations of ϕ and ψ respectively. Then $E_i \cap E_k = \emptyset$, for all $i \le i \neq k \le n$, $F_j \cap F_l = \emptyset$, for all $1 \le j \neq l \le m$, $X = \cup_{i=1}^{n} E_i$, and $X = \cup_{j=1}^{m} F_j$.

[1] For $1 \le i \le n$ and $1 \le j \le m$, let $G_{ij} = E_i \cap F_j$. Then, for each $x \in G_{ij}$, $\phi(x) + \psi(x) = a_i + b_j$, i.e., the function $\phi + \psi$ takes the values $a_i + b_j$ on $E_i \cap F_j$. For all $1 \le i, k \le n$ and $1 \le j, l \le m$,

$$\begin{aligned}G_{ij} \cap G_{kl} &= (E_i \cap F_j) \cap (E_k \cap F_l) \\ &= (E_i \cap E_k) \cap (F_j \cap F_l) \\ &= \emptyset \cap \emptyset = \emptyset.\end{aligned}$$

That is, $\{G_{ij} : 1 \le i \le n \text{ and } 1 \le j \le m\}$ is a collection of nm pairwise disjoint sets. Furthermore,

$$E_i = E_i \cap X = E_i \cap \left(\bigcup_{j=1}^{m} F_j\right) = \bigcup_{j=1}^{m}(E_i \cap F_j) = \bigcup_{j=1}^{m} G_{ij},$$

$$F_j = F_j \cap X = F_j \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n}(E_i \cap F_j) = \bigcup_{i=1}^{n} G_{ij}, \text{ and}$$

$$X = X \cap X = \left(\bigcup_{i=1}^{n} E_i\right) \cap \left(\bigcup_{j=1}^{m} F_j\right) = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}(E_i \cap F_j) = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m} G_{ij}$$

Therefore,

$$\phi + \psi = \sum_{i=1}^{n}\sum_{j=1}^{m}(a_i + b_j)\chi_{G_{ij}}.$$

With this representation of $\phi + \psi$, the numbers $a_i + b_j$ are not necessarily distinct. Let $c_1, c_2, \ldots, c_k$ be the distinct values assumed by $\phi + \psi$. Then

$$\begin{aligned}
\int_X (\phi + \psi)d\mu &= \sum_{r=1}^{k} c_r \mu(\{x : \phi(x) + \psi(x) = c_r\}) \\
&= \sum_{r=1}^{k} c_r \mu\left(\bigcup_{a_i+b_j=c_r} E_i \cap F_j\right) \\
&= \sum_{r=1}^{k} c_r \sum_{a_i+b_j=c_r} \mu(E_i \cap F_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}(a_i + b_j)\mu(G_{ij}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i\mu(G_{ij}) + \sum_{i=1}^{n}\sum_{j=1}^{m} b_j\mu(G_{ij}) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{m}\mu(G_{ij}) + \sum_{i=1}^{n} b_j \sum_{j=1}^{m}\mu(G_{ij}) \\
&= \sum_{i=1}^{n} a_i\mu(E_i) + \sum_{j=1}^{m} b_j\mu(F_j) \\
&= \int_X \phi d\mu + \int_X \psi d\mu.
\end{aligned}$$

[2] If $c = 0$, then $c\phi$ vanishes identically and hence, $\int_X c\phi d\mu = c\int_X \phi d\mu$. Assume that $c > 0$. If $\phi = \sum_{i=1}^{n} a_i\chi_{E_i}$ is the canonical representation of ϕ, then $c\phi = \sum_{i=1}^{n} ca_i\chi_{E_i}$ is the canonical representation of $c\phi$. Therefore,

$$\int_X c\phi d\mu = \sum_{i=1}^{n} ca_i\mu(E_i) = c\sum_{i=1}^{n} a_i\mu(E_i) = c\int_X \phi d\mu.$$

[3] As shown above,

$$\int_X \phi = \sum_{i=1}^{n} a_i \mu(E_i) = \sum_{i=1}^{n} a_i \mu\left(\bigcup_{j=1}^{m} E_i \cap F_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \mu(E_i \cap F_j), \text{ and}$$

$$\int_X \psi = \sum_{j=1}^{m} b_j \mu(F_j) = \sum_{j=1}^{m} b_j \mu\left(\bigcup_{i=1}^{n} F_j \cap E_i\right) = \sum_{j=1}^{m}\sum_{i=1}^{n} b_j \mu(E_i \cap F_j).$$

Now suppose that $\phi \le \psi$. Then, for each $x \in E_i \cap F_j$, $a_i = \phi(x) \le \psi(x) = b_j$. i.e., $a_i \le b_j$, for all i, j such that $E_i \cap F_j \ne \emptyset$. It follows that $\int_X \phi d\mu \le \int_X \psi d\mu$.

[4] Let ϕ be as above. Then

$$\begin{aligned}
\int_{A\cup B} \phi d\mu &= \sum_{i=1}^{n} a_i \mu((A \cup B) \cap E_i) \\
&= \sum_{i=1}^{n} a_i \mu[(A \cap E_i) \cup (B \cap E_i)] \\
&= \sum_{i=1}^{n} a_i [\mu(A \cap E_i) + \mu(B \cap E_i)] \\
&= \sum_{i=1}^{n} a_i \mu(A \cap E_i) + \sum_{i=1}^{n} \mu(B \cap E_i) \\
&= \int_A \phi d\mu + \int_B \phi d\mu.
\end{aligned}$$

[5] We have that, for $A \in \Sigma$,

$$\nu(A) = \int_A \phi d\mu = \sum_{i=1}^{n} a_i \mu(A \cap E_i).$$

If $A = \emptyset$, then $A \cap E_i = \emptyset$, for each $i = 1, 2, \ldots, n$. Hence, $\mu(A \cap E_i) = 0$, for $i = 1, 2, \ldots, n$ and so $\nu(\emptyset) = 0$.

Let (A_k) be a sequence of pairwise disjoint measurable sets and $A = \bigcup_{k=1}^{\infty} A_k$. Then, for each $i = 1, 2, \ldots, n$,

$$E_i \cap A = E_i \cap \left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty} (E_i \cap A_k), \quad \text{a disjoint union.}$$

Thus,

$$\begin{aligned}\nu(A) &= \sum_{i=1}^{n}\sum_{i=1}^{n} a_i \mu(A \cap E_i) \\ &= \sum_{i=1}^{n} a_i \mu\left(\bigcup_{k=1}^{\infty} E_i \cap A_k\right) \\ &= \sum_{i=1}^{n} a_i \sum_{k=1}^{\infty} \mu(E_i \cap A_k) \\ &= \sum_{k=1}^{\infty}\sum_{i=1}^{n} a_i \mu(E_i \cap A_k) \\ &= \sum_{k=1}^{\infty} \nu(A_k).\end{aligned}$$

That is, $\nu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \nu(A_k)$.

[6] We have by [5], that

$$0 \leq \int_A \phi d\mu = \sum_{i=1}^{n} a_i \mu(A \cap E_i) \leq \sum_{i=1}^{n} a_i \mu(A) = 0.$$

Thus, $\int_A \phi d\mu = 0$.

□

The next result asserts that changing a simple function on a null set does not change the integral.

3.3.3 Corollary

Let (X, Σ, μ) be a measure space and ϕ a nonnegative simple function. If A and B are measurable sets such that $A \subseteq B$ and $\mu(B \backslash A) = 0$, then

$$\int_A \phi d\mu = \int_B \phi d\mu.$$

PROOF.

Noting that $B = A \cup (B \backslash A)$, a disjoint union, we have by Proposition 3.3.2 [4] and [6], that

$$\int_B \phi d\mu = \int_{A\cup(B\backslash A)} \phi d\mu = \int_A \phi d\mu + \int_{B\backslash A} \phi d\mu = \int_A \phi d\mu + 0 = \int_A \phi d\mu.$$

□

3.3.2 Integral of a nonnegative measurable function

We showed in Theorem 3.2.8 that every nonnegative measurable function is a pointwise limit of an increasing sequence of of nonnegative simple functions. Using this fact, we are able to define the integral of a nonnegative measurable function using the integrals of nonnegative simple functions.

3.3.4 Definition
Let f be a nonnegative measurable function. The **Lebesgue integral** of f with respect to μ, denoted by $\int_X f d\mu$, is defined as

$$\int_X f d\mu = \sup\left\{\int_X \phi d\mu : 0 \leq \phi \leq f, \phi \text{ is a simple function}\right\}.$$

If $E \in \Sigma$, then we define the **Lebesgue integral of f over E with respect to** μ as

$$\int_E f d\mu = \int f\chi_E d\mu.$$

Notice that if f is a nonnegative simple function, then Definitions 3.3.1 and 3.3.4 coincide.

3.3.5 Proposition
Let f and g be nonnegative measurable functions and c a nonnegative real number.

[1] $\int_X cf d\mu = c\int_X f d\mu$.

[2] If $f \leq g$, then $\int_X f d\mu \leq \int_X g d\mu$.

[3] If A and B are measurable sets such that $A \subseteq B$, then

$$\int_A f d\mu \leq \int_B f d\mu.$$

PROOF.

[1] If $c = 0$, then the equality holds trivially. Assume that $c > 0$. Then

$$\begin{aligned}\int_X cf d\mu &= \sup\left\{\int_X \phi d\mu : 0 \leq \phi \leq cf, \phi \text{ a simple function}\right\}\\ &= \sup\left\{c\int_X \frac{\phi}{c} d\mu : 0 \leq \frac{\phi}{c} \leq f, \phi \text{ a simple function}\right\}\\ &= c\sup\left\{\int_X \frac{\phi}{c} d\mu : 0 \leq \frac{\phi}{c} \leq f, \phi \text{ a simple function}\right\}\\ &= c\int_X f\delta\mu.\end{aligned}$$

[2] Since $f \leq g$, it follows that $\{\phi : 0 \leq \phi \leq f, \phi \text{ a simple function}\} \subset \{\phi : 0 \leq \phi \leq g, \phi \text{ a simple function}\}$. Thus,

$$\int_X f d\mu = \sup_{0\leq\phi\leq f}\int_X \phi d\mu \leq \sup_{0\leq\phi\leq g}\int_X \phi d\mu = \int_X g d\mu.$$

[3] If $A \subseteq B$, then $\chi_A \leq \chi_B$. Therefore, for any nonnegative measurable function f, we have that $f\chi_A \leq f\chi_B$. By [2], it follows that

$$\int_A f d\mu = \int_X f\chi_A d\mu \leq \int_X f\chi_B d\mu = \in_B f d\mu.$$

Conversely, assume that $f = 0$ a.e. on X. Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ be a nonnegative simple function, written in canonical form, such that $\phi \leq f$. Then $\phi = 0$ a.e. on X. Since $E_i = \{x \in X : \phi(x) = a_i, i = 1, 2, \ldots, n\}$, we have that $\mu\left(\bigcup_{i=1}^{n} E_i\right) = 0$. But since

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i),$$

we have that $\mu(E_i) = 0$, for each $i = 1, 2, \ldots, n$. Therefore,

$$\int_X \phi d\mu = \sum_{i=1}^{n} a_i \mu(E_i) = 0.$$

It then follows that

$$\int_X f d\mu = \sup_{0 \leq \phi \leq f} \int_X \phi d\mu = 0.$$

□

3.3.8 Proposition
Let (X, Σ, μ) be a measure space and f a nonnegative measurable function. If $A \in \Sigma$ and $\mu(A) = 0$, then

$$\int_A f d\mu = 0.$$

PROOF.

Let ϕ be a nonnegative simple function such that $\phi \leq f$. Then, by Proposition 3.3.2 [6], we have that $\int_A \phi d\mu = 0$. Therefore

$$\int_A f d\mu = \sup_{0 \leq \phi \leq f} \int_A \phi d\mu = \sup_{0 \leq \phi \leq f} \{0\} = 0.$$

□

3.3.9 Theorem (Monotone convergence theorem)
Let (X, Σ, μ) be a measure space, (f_n) a sequence of measurable functions on X such that $0 \leq f_n \leq f_{n+1}$, for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = f(x)$, for each $x \in X$. Then f is measurable and

$$\int_X f d\mu = \int_X \left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} \int_X f_n d\mu.$$

PROOF.

We have already shown in Corollary 3.2.5 that f is measurable.
Since $0 \leq f_n \leq f_{n+1} \leq f$, for each $n \in \mathbb{N}$, it follows, by Proposition 3.3.5, that

$$\int_X f_n d\mu \leq \int_X f_{n+1} d\mu \leq \int_X f d\mu,$$

for each $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \int_X f_n d\mu \leq \int_X f d\mu \tag{3.3}$$

We now prove the reverse inequality. Let ϕ be a simple function such that $0 \leq \phi \leq f$. Choose and fix ϵ such that $0 < \epsilon < 1$. For each $n \in \mathbb{N}$, let

$$A_n = \{x \in X : f_n(x) \geq (1-\epsilon)\phi(x)\}.$$

Now, for each $n \in \mathbb{N}$, A_n is measurable, and since (f_n) is an increasing sequence, we have that $A_n \subseteq A_{n+1}$. Furthermore $X = \bigcup_{n=1}^{\infty} A_n$. Since for each $n \in \mathbb{N}$, $A_n \subset X$, it follows that $\bigcup_{n=1}^{\infty} A_n \subseteq X$. For the reverse containment, let $x \in X$. If $f(x) = 0$, then $\phi(x) = 0$, and therefore $x \in A_1$. If on the other hand, $f(x) > 0$, then $(1-\epsilon)\phi(x) < f(x)$. Since $f_n \uparrow f$ and $0 < \epsilon < 1$, there is a natural number N such that $(1-\epsilon)\phi(x) \leq f_n(x)$, for all $n \geq N$. Thus, $x \in A_n$, for all $n \geq N$, and so, $X \subseteq \bigcup_{n=1}^{\infty} A_n$. Now, for each $n \in \mathbb{N}$,

$$\int_X f_n d\mu \geq \int_{A_n} f_n d\mu \geq (1-\epsilon)\int_{A_n} \phi d\mu. \tag{3.4}$$

Define $\nu : \Sigma \to [0,\infty]$ by

$$\nu(E) = \int_E \phi d\mu, \quad E \in \Sigma.$$

We have already shown in Proposition 3.3.2 [5], that ν is a measure on Σ. Since (A_n) is an increasing sequence of measurable sets such that $X = \bigcup_{n=1}^{\infty} A_n$, we have, by Theorem 2.3.6 [3], that

$$\nu(X) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n).$$

That is, $\int_X \phi d\mu = \lim_{n\to\infty} \int_{A_n} \phi d\mu$. Letting $n \to \infty$ in (3.4), we have that

$$\lim_{n\to\infty} \int_X f_n d\mu \geq (1-\epsilon)\int_X \phi d\mu.$$

Since ϵ is arbitrary, it follows that

$$\lim_{n\to\infty} \int_X f_n d\mu \geq \int_X \phi d\mu,$$

for any nonnegative simple function ϕ such that $\phi \leq f$. Then,

$$\lim_{n\to\infty} \int_X f_n d\mu \geq \sup\left\{\int_X \phi d\mu : \phi \text{ is simple and } 0 \leq \phi \leq f\right\} = \int_X f d\mu. \tag{3.5}$$

From (3.3) and (3.5), we have that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

□

3.3.10 Corollary

Let f and g be nonnegative measurable functions. Then

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

PROOF.

By Theorem 3.2.8, there are monotonic increasing sequences (ϕ_n) and (ψ_n) of nonnegative simple functions such that $\phi_n \to f$ and $\psi_n \to g$, as $n \to \infty$. Then $(\phi_n + \psi_n)$ is a monotonic increasing sequence which converges to $f + g$. By the Monotone Convergence Theorem (Theorem 3.3.9), we have that

$$\begin{aligned}
\int_X (f+g)d\mu &= \lim_{n\to\infty} \int_X (\phi_n + \psi_n)d\mu \\
&= \lim_{n\to\infty} \left[\int_X \phi_n d\mu + \int_X \psi_n d\mu \right] \\
&= \lim_{n\to\infty} \int_X \phi_n d\mu + \lim_{n\to\infty} \int_X \psi_n d\mu \\
&= \int_X f d\mu + \int_X g d\mu \quad \text{(by the Monotone Convergence Theorem)}
\end{aligned}$$

□

3.3.11 Corollary

Let (X, Σ, μ) be a measure space and (f_n) a sequence of nonnegative measurable functions on X. Then, for each $n \in \mathbb{N}$,

$$\int_X \sum_{k=1}^{n} f_k d\mu = \sum_{k=1}^{n} \int_X f_k d\mu.$$

PROOF.

(Induction on n). If $n = 1$, then the result is obviously true. If $n = 2$, then, by Corollary 3.3.10, the result holds. Assume that

$$\int_X \sum_{k=1}^{n-1} f_k d\mu = \sum_{k=1}^{n-1} \int_X f_k d\mu.$$

then

$$\begin{aligned}
\int_X \sum_{k=1}^{n} f_k d\mu &= \int_X \left(\sum_{k=1}^{n-1} f_k + f_n \right) d\mu \\
&= \int_X \sum_{k=1}^{n-1} f_k + \int_X f_n d\mu \quad \text{(by Corollary 3.3.10)} \\
&= \sum_{k=1}^{n-1} \int_X f_k d\mu + \int_X f_n d\mu \quad \text{(by the induction hypothesis)} \\
&= \sum_{k=1}^{n} \int_X f_k d\mu.
\end{aligned}$$

□

3.3.12 Corollary (Beppo Levi)

Let (X, Σ, μ) be a measure space (f_n) a sequence of nonnegative measurable functions on X and $f =$

PROOF.

Since $f_n \rightarrow^{\text{a.e.}}$, there is a set $N \in \Sigma$ such that $\mu(N) = 0$ and $f_n \rightarrow f$ pointwise on N^c. By Theorem 3.3.9,

$$\lim_{n\to\infty} \int_{N^c} f_n d\mu = \int_{N^c} f d\mu.$$

From Corollary 3.3.13, it follows that

$$\int_X f d\mu = \int_{N^c} f d\mu = \lim_{n\to\infty} \int_{N^c} f_n d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

□

3.3.15 Proposition
Let (X, Σ, μ) be a measure space and f a nonnegative measurable function on X. Then the set function $\lambda : \Sigma \to [0, \infty]$ defined by

$$\lambda(A) = \int_A f d\mu, \quad A \in \Sigma$$

is a measure on X. Furthermore, for any nonnegative measurable function g,

$$\int_X g d\lambda = \int_X g f d\mu.$$

PROOF.

We note that

$$\lambda(A) = \int_A f d\mu = \int_X \chi_A f d\mu.$$

Since $f \geq 0$, it follows that $\lambda(A) \geq 0$, for each $A \in \Sigma$. If $A = \emptyset$, then $\chi_A = 0$ and so $\chi_A f = 0$. Therefore

$$\lambda(\emptyset) = \lambda(A) = \int_X \chi_A f d\mu = \int_X 0 d\mu = 0.$$

Let (A_n) be a sequence of pairwise disjoint measurable sets and $A = \bigcup_{n=1}^{\infty} A_n$. Then

$$\chi_A = \chi_{\bigcup_{n=1}^{\infty} A_n} = \sum_{n=1}^{\infty} \chi_{A_n}.$$

Therefore, $\chi_A f = \sum_{n=1}^{\infty} \chi_{A_n} f$, and so

$$\begin{aligned}
\lambda(A) &= \int_X \chi_A f d\mu = \int_X \sum_{n=1}^{\infty} \chi_{A_n} f d\mu \\
&= \sum_{n=1}^{\infty} \int_X \chi_{A_n} f d\mu \quad \text{(by Corollary 3.3.12)} \\
&= \sum_{n=1}^{\infty} \int_{A_n} f d\mu \\
&= \sum_{n=1}^{\infty} \lambda(A_n).
\end{aligned}$$

That is, $\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n)$.

If $g = \chi_A$, for some $A \in \Sigma$, i.e., g is a characteristic function, then

$$\int\limits_X g d\lambda = \int\limits_X \chi_A d\lambda = \int\limits_A d\lambda = \lambda(A)$$

and

$$\int\limits_X g f d\mu = \int\limits_X \chi_A f d\mu = \int\limits_A f d\mu = \lambda(A).$$

Thus, $\int_X g d\lambda = \int_X g f d\mu$. Assume that $g = \phi$, a nonnegative simple function and let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ be the canonical representation of ϕ. Then,

$$\begin{aligned}
\int\limits_X g d\lambda = \int\limits_X \phi d\lambda = \int\limits_X \left(\sum_{i=1}^{n} a_i \chi_{E_i}\right) d\lambda \\
&= \sum_{i=1}^{n} a_i \int\limits_X \chi_{E_i} d\lambda \quad \text{(by Corollary 3.3.11 and Proposition 3.3.5)} \\
&= \sum_{i=1}^{n} a_i \int\limits_{E_i} d\lambda \\
&= \sum_{i=1}^{n} a_i \lambda(E_i),
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_X g f d\mu = \int\limits_X \phi f d\mu = \int\limits_X \left(\sum_{i=1}^{n} a_i \chi_{E_i}\right) f d\mu \\
&= \sum_{i=1}^{n} a_i \int\limits_X \chi_{E_i} f d\mu \quad \text{(by Corollary 3.3.11 and Proposition 3.3.5)} \\
&= \sum_{i=1}^{n} a_i \int\limits_{E_i} f d\mu \\
&= \sum_{i=1}^{n} a_i \lambda(E_i).
\end{aligned}$$

Therefore, if g is a nonnegative simple function, then $\int_X g d\lambda = \int_X g f d\mu$.

Let g be a nonnegative measurable function. Then, by Theorem 3.2.8, there is an increasing sequence (ϕ_n) of nonnegative simple functions such that $\phi_n \to^{n\to\infty} g$. Then $\phi_n f \to^{n\to\infty} gf$. Clearly the $\phi_n f$ are nonnegative and measurable functions for each $n \in \mathbb{N}$ (see Proposition 3.2.4). By the Monotone Convergence Theorem (Theorem 3.3.9), we have that

$$\begin{aligned}
\int\limits_X g d\lambda &= \lim_{n\to\infty} \int\limits_X \phi_n d\lambda \\
&= \lim_{n\to\infty} \int\limits_X \phi_n f d\mu \\
&= \int\limits_X g f d\mu.
\end{aligned}$$

If $f < 0$, then $f^+ = 0$ and so, for any $x \in X$,

$$f^+(x) = 0 \leq g(x).$$

A similar argument shows that $f^- \leq h$.

3.3.17 Definition
Let (X, Σ, μ) be a measure space and f a measurable function on X. If $\int_X f^+ d\mu < \infty$ or $\int_X f^- d\mu < \infty$, then we define the **integral** of f, denoted by $\int_X f d\mu$, as the extended real number

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

If $\int_X f^+ d\mu < \infty$ and $\int_X f^- d\mu < \infty$, then f is said to be **(Lebesgue) integrable on** X. The set of all integrable functions is denoted by $\mathcal{L}^1(X, \mu)$.

It is clear that f is integrable if and only if $\int_X |f| d\mu < \infty$. In this case,

$$\int_X |f| d\mu = \int_X f^+ d\mu + \int_X f^- d\mu.$$

3.3.18 Remark
Let (X, Σ, μ) be a measure space and f a measurable function on X. If f is integrable on X, then $|f| < \infty$ a.e. on X. Indeed, if $A \in \Sigma$ with $\mu(A) > 0$ and $|f| = \infty$ on A, then, for each $n \in \mathbb{N}$, $|f| > n\chi_A$. Hence,

$$\int_X |f| d\mu \geq \int_X n\chi_A d\mu = n \int_A d\mu = n\mu(A).$$

Letting n tend to infinity, we have that $\int_X |f| d\mu = \infty$.

3.3.19 Theorem
Let (X, Σ, μ) be a measure space, $f, g \in \mathcal{L}^1(X, \mu)$ and let $c \in \mathbb{R}$. Then

[1] $cf \in \mathcal{L}^1(X, \mu)$ and $\int_X (cf) d\mu = c \int_X f d\mu$.

[2] $f + g \in \mathcal{L}^1(X, \mu)$ and $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$.

PROOF.

[1] Assume that $c \geq 0$. Then,

$$(cf)^+ = \max\{cf, 0\} = c \max\{f, 0\} = c(f^+) \text{ and}$$
$$(cf)^- = \max\{-(cf), 0\} = c \max\{-f, 0\} = c(f^-)$$

Therefore,

$$\int_X (cf)^+ d\mu = \int_X cf^+ d\mu = c \int_X f^+ d\mu < \infty \text{ and}$$
$$\int_X (cf)^- d\mu = \int_X cf^- d\mu = c \int_X f^- d\mu < \infty.$$

Hence, cf is integrable and

$$\begin{aligned}\int_X (cf)d\mu &= \int_X (cf)^+ d\mu - \int_X (cf)^- d\mu \\ &= c\int_X f^+ d\mu - c\int_X f^- d\mu \\ &= c\Big(\int_X f^+ d\mu - \int_X f^- d\mu\Big) \\ &= c\int_X f d\mu.\end{aligned}$$

If $c < 0$, then $c = -k$ for some $k > 0$. Therefore

$$(cf)^+ = \max\{cf, 0\} = \max\{-kf, 0\} = k\max\{-f, 0\} = kf^- = -cf^- \text{ and}$$
$$(cf)^- = \max\{-(cf), 0\} = \max\{-cf, 0\} = \max\{kf, 0\} = k\max\{f, 0\} = kf^+ = -cf^+.$$

From this it follows that

$$\int_X (cf)^+ d\mu = \int_X (-cf^-)d\mu = -c\int_X f^- d\mu < \infty \text{ and}$$
$$\int_X (cf)^- d\mu = \int_X (-cf^+)d\mu = -c\int_X f^+ d\mu < \infty.$$

Therefore, cf is integrable and

$$\begin{aligned}\int_X (cf)d\mu &= \int_X (cf)^+ d\mu - \int_X (cf)^- d\mu \\ &= -c\int_X f^- d\mu - (-c)\int_X f^+ d\mu \\ &= -c\int_X f^- d\mu + c\int_X f^+ d\mu \\ &= c\Big(\int_X f^+ d\mu - \int_X f^- d\mu\Big) \\ &= c\int_X f d\mu.\end{aligned}$$

[2] Since

$$f + g = (f+g)^+ - (f+g)^- \text{ and}$$
$$f + g = (f^+ - f^-) + (g^+ - g^-) = (f^+ g^+) - (f^- + g^-),$$

we have that

$$(f+g)^+ \leq f^+ g^+ \text{ and } (f+g)^- \leq f^- + g^-.$$

By Proposition 3.3.5 [2], it follows that

$$\int_X (f+g)^+ d\mu \leq \int_X (f^+ + g^+) d\mu \leq \int_X f^+ d\mu + \int_X g^+ d\mu < \infty \text{ and}$$
$$\int_X (f+g)^- d\mu \leq \int_X (f^- + g^-) d\mu \leq \int_X f^- d\mu + \int_X g^- d\mu < \infty.$$

Thus, $f+g$ is integrable. Since $(f+g)^+ - (f+g)^- = f+g = f^+ - f^- + g^+ - g^-$, we have that

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

By Corollary 3.3.10, it follows that

$$\begin{aligned}
&\int_X [(f+g)^+ + f^- + f^-] d\mu = \int_X [(f+g)^- + f^+ + g^+] d\mu \\
\Leftrightarrow &\int_X (f+g)^+ d\mu + \int_X f^- d\mu + \int_X g^- d\mu = \int_X (f+g)^- d\mu + \int_X f^+ d\mu + \int_X g^+ d\mu \\
\Leftrightarrow &\int_X (f+g)^+ d\mu - \int_X (f+g)^- d\mu = \int_X f^+ d\mu + \int_X f^- d\mu + \int_X g^+ d\mu - \int_X g^- d\mu \\
\Leftrightarrow &\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.
\end{aligned}$$

□

3.3.20 Corollary
Let (X, Σ, μ) be a measure space. Then $\mathcal{L}^1(X, \mu)$ is a vector space with respect to the usual operations of addition and scalar multiplication.

3.3.21 Proposition
Let (X, Σ, μ) be a measure space, $f, g \in \mathcal{L}^1(X, \mu)$. If $f \leq g$ a.e. on X, then

$$\int_X f d\mu \leq \int_X g d\mu.$$

PROOF.

Since $g - f \geq 0$ a.e. on X, we have that

$$\int_X g d\mu - \int_X f d\mu = \int_X (g-f) d\mu \geq 0.$$

It now follows that $\int_X f d\mu \leq \int_X g d\mu$.

□

3.3.22 Lemma
Let (X, Σ, μ) be a measure space. If g is a Lebesgue integrable function on X and f is a measurable function such that $|f| \leq g$ a.e., then f is Lebesgue integrable on X.

PROOF.

Since $|f| = f^+ + f^-$, it follows that $f^+ \leq g$ and $f^- \leq g$. Hence,

$$0 \leq \int_X f^+ d\mu \leq \int_X g d\mu \text{ and } 0 \leq \int_X f^- d\mu \leq \int_X g d\mu.$$

Therefore f^+ and f^- are both integrable.

□

The following theorem, known as Lebesgue's Dominated Convergence Theorem, provides a useful criterion for the interchange of limits and integrals and is of fundamental importance in measure theory. As a motivation for the measure theoretic approach to integration, think about how careful one must be when interchanging the limits and integrals in Riemann integration.

3.3.23 Theorem (Lebesgue's Dominated Convergence Theorem)

Let (X, Σ, μ) be a measure space, (f_n) a sequence of measurable functions on X such that $f_n \to^{n\to\infty} f$ a.e., for f a measurable function. If there is a Lebesgue integrable function g such that for each $n \in \mathbb{N}$, $|f_n| \leq g$ a.e., then f is Lebesgue integrable and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

PROOF.

Since $|f_n| \leq g$ a.e., for each $n \in \mathbb{N}$, it follows that $|f| \leq g$ a.e. By Lemma 3.3.22, we have that f is Lebesgue integrable.

Since $g \pm f_n \geq 0$, for each $n \in \mathbb{N}$, we have by Fatou's Lemma (Theorem 3.3.16), that

$$\int_X g d\mu + \int_X f d\mu = \int_X (g + f) d\mu \leq \liminf_n \int_X (g + f_n) d\mu, \text{ and} \tag{3.8}$$

$$\int_X g d\mu - \int_X f d\mu = \int_X (g - f) d\mu \leq \liminf_n \int_X (g - f_n) d\mu. \tag{3.9}$$

Now,

$$\liminf_n \int_X (g + f_n) d\mu = \int_X g d\mu + \liminf_n \int_X f_n d\mu, \text{ and} \tag{3.10}$$

$$\liminf_n \int_X (g - f_n) d\mu = \int_X g d\mu - \limsup_n \int_X f_n d\mu. \tag{3.11}$$

Since g is Lebesgue integrable, we have, from (3.8) and (3.10), that

$$\int_X f d\mu \leq \liminf_n \int_X f_n d\mu. \tag{3.12}$$

Similarly, from (3.9) and (3.11), we have that

$$\limsup_n \int_X f_n d\mu \leq \int_X f d\mu \tag{3.13}$$

From (3.12) and (3.13), it follows that

$$\limsup_n \int_X f_n d\mu \leq \int_X f d\mu \leq \liminf_n \int_X f_n d\mu,$$