Nonlinear Dynamical System

I) continuous-time Dynamical System:-

1) one-Dimensional system:-

a) Approximate solution and the asymptotic behavior:-

consider the dynamical system $\frac{dx}{dt} = f(x)$, where $f(x)$ is a nonlinear function

let x^* be an equilibrium point $f(x^*) = 0$

Using Taylor series around x^*, then

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2}(x - x^*)^2 + \cdots$$

The linear term:-

$f(x^*) + f'(x^*)(x - x^*)$ is called the linearization (tangent line approximation) to the nonlinear function $f(x)$ in the neighborhood of x^*.

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$f(x^*) = 0$ $\qquad\qquad \therefore f(x) = f'(x^*)(x - x^*)$

The dynamical system become $\frac{d(x-x^*)}{dt} = f'(x^*)(x - x^*)$

Let $y = x - x^*$ $\qquad\qquad \therefore \frac{dy}{dt} = f'(x^*)y$

$$\int \frac{dy}{y} = f'(x^*) \int dt \qquad \Rightarrow \qquad \ln y = f'(x^*)t + c$$

$$y = ce^{f'(x^*)t} \qquad \Rightarrow \qquad x - x^* = ce^{f'(x^*)t}$$

The approximate solution for the nonlinear dynamical system

$$x(t) = ce^{f'(x^*)t} + x^*$$

The asymptotic behavior:-

As $t \to \infty$

If $f'(x^*) < 0$ $\Rightarrow$ $e^{f'(x^*)t} \to 0$

If $f'(x^*) > 0$ $\Rightarrow$ $e^{f'(x^*)t} \to \infty$ $\Rightarrow$ $x(t) \to \infty$

If $f'(x^*) = 0$ $\Rightarrow$ $e^{f'(x^*)t} \to finite$

b) Linear stability analysis:-

let x^* be an equilibrium point of the dynamical system $x' = f(x)$

If $f'(x^*) < 0$ $\Rightarrow$ x^* is stable

If $f'(x^*) > 0$ $\Rightarrow$ x^* is unstable

If $f'(x^*) = 0$ $\Rightarrow$ test fails

x^* may be stable or unstable

According to nonlinear term (test fails)

Example:- determined the equilibrium and their linearized stability of the dynamical system $x' = x^2 - 1$

Solution:- to determine the equilibrium point $x' = 0$

$x^2 - 1 = 0$ $x_1^* = 1,$ $x_2^* = -1$

$$f'(x) = -2x$$

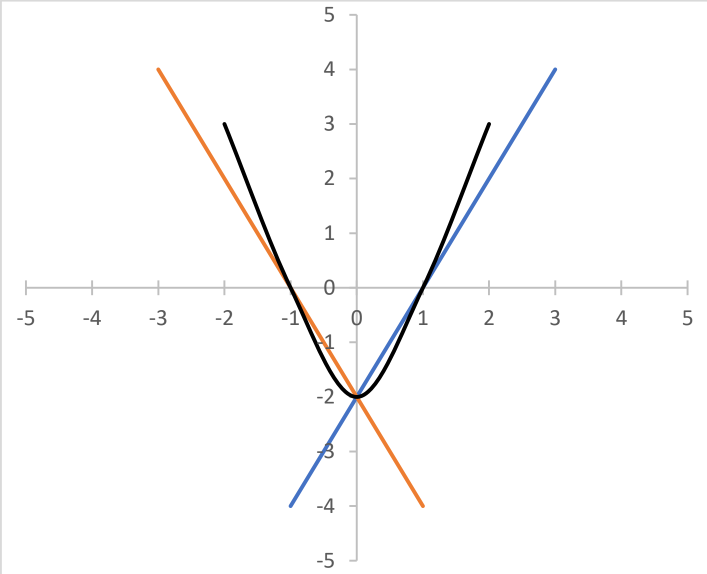
$f'(x_1^*) = 2 > 0$ unstable, $f'(x_2^*) = -2 < 0$ stable

The linearization around $f(x)$

$f(x) = f'(x^*)(x - x^*)$

Around $x_1^* = 1 \qquad y = 2x - 2$

Around $x_2^* = -1 \quad y = -2x - 2$



Example:- determined the equilibrium and their linearized stability of the dynamical system $x' = x - x^3$

Solution:- to determine the equilibrium $x' = 0$

$$x - x^3 = 0 \qquad \Rightarrow \qquad x(1 - x^2) = 0$$

$$x_1^* = 0 \qquad x_2^* = 1 \qquad x_2^* = -1$$

$$f'(x_1^*) = 1 > 0 \quad \text{unstable,} \qquad f'(x_2^*) = f'(x_3^*) = -2 < 0 \quad \text{stable}$$

The linearization of $f(x)$

Around $x_1^* = 0 \qquad \Rightarrow \qquad y = x$

Around $x_2^* = 1 \qquad \Rightarrow \qquad y = -2x + 2$

Around $x_3^* = -1 \quad \Rightarrow \qquad y = -2x - 2$

2) Two-Dimensional system:-

Consider a two-dimensional system $\frac{dx}{dt} = f(x)$, $f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$ is a nonlinear function

Let χ^* be an equilibrium point

The nonlinear function $f(x)$ is approximate by the linear function $f(x) \approx A(x - x^*)$ where A is the Jacobin of $f(x)$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

Linearization and stability analysis:-

Real eigenvalue

If $\lambda_1, \lambda_2 < 0$ $\Rightarrow$ stable node

If $\lambda_1, \lambda_2 > 0$ $\Rightarrow$ unstable node

If λ_1, λ_2 have opposite signs $\Rightarrow$ saddle point

If $\lambda_i \leq 0,\ \lambda_j = 0$ $\Rightarrow$ Test fails

Complex eigenvalue $\lambda = \alpha + i\beta$

If $\alpha < 0$ $\Rightarrow$ stable spiral

If $\alpha > 0$ $\Rightarrow$ unstable spiral

If $\alpha = 0$ $\Rightarrow$ Test fails

Example:- $\frac{dx}{dt} = \begin{bmatrix} -x + xy \\ -4y + 8y \end{bmatrix}$

Solution:- to determine the equilibrium $f(x) = 0$

$-x + xy = 0 \quad \Rightarrow \quad x(-1 + y) = 0 \quad \Rightarrow \quad x = 0$ and $y = 1$

$-4y + 8xy = 0 \quad \Rightarrow \quad y(-4 + 8x) = 0 \quad \Rightarrow \quad x = \frac{1}{2}$ and $y = 0$

The equilibrium $\quad x_1^* = \begin{bmatrix}0\\0\end{bmatrix} \quad x_2^* = \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}$

The Jacob matrix

$$A = \begin{bmatrix}-1+y & x\\ 8y & -4+8x\end{bmatrix}$$

$A(x_1^*) = \begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix} \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = -4$

$x_1^* = \begin{bmatrix}0\\0\end{bmatrix}$ is a stable node

$A(x_2^*) = \begin{bmatrix}0 & \frac{1}{2}\\ 8 & 0\end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 2, \quad \lambda_2 = -2$

$x_2^* = \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}$ is a saddle node

II) Discrete-time Dynamical System:-

1) one-Dimensional system:-

a) Approximate solution and the asymptotic behavior:-

consider the dynamical system $x_{k+1} = f(x_k), \;\; x(0) = x_0$, where $f(x_k)$ is a nonlinear function.

Let x^* be an equilibrium point $f(x^*) = x^*$

Using the Taylor series expansion around x^*, then

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2}(x - x^*)^2 + \cdots$$

$$= f'(x^*)x + (1 - f'(x^*))x^*$$

Comparing with the equation $f(x) = ax + b$

$$a = f'(x^*) \qquad b = (1 - f'(x^*))x^*$$

The approximate solution

$$x(k) = \begin{cases} (f'(x^*))^k x + (1 - f'(x^*))^k x^* & , f'(x^*) \neq 1 \\ \\ x_0 + k(1 - f'(x^*))x^* & , f'(x^*) = 1 \end{cases}$$

The asymptotic behavior:-

As $k \to \infty$

If $|f'(x^*)| < 1$ $\quad f'(x^*))^k \to 0 \quad \Rightarrow \quad x(k) \to x^*$

If $|f'(x^*)| > 1$ $\quad f'(x^*))^k \to \infty \quad \Rightarrow \quad x(k) \to \infty$

If $f'(x^*) = 1$ $\quad x(k) \to x_0$

$$\text{If } f'(x^*) = -1 \quad \begin{cases} -x_0 + 2x^* & ,k = 1,3,5,\cdots \\ \\ x_0 & ,k = 2,4,6,\cdots \end{cases}$$

Example:- prove that 2 is an equilibrium point of the dynamical system $x_{k+1} = x_k^3 - x_k^2 - 2$, $x_0 = 3$. Find an approximate solution of the system around 2, and discuss the asymptotic behavior.

Solution:- $\quad f(x) = x^3 - x^2 - 2$

$$f(2) = 8 - 4 - 2 = 2 \quad \Rightarrow \quad \therefore 2 \text{ is an equilibrium point}$$

$$f'(x) = 3x^2 - 2x \quad \Rightarrow \quad f'(2) = 12 - 4 = 8$$

The approximate solution

$$x(k) = (f'(x^*))^k x + \left(1 - f'(x^*)\right)^k x^* = 3(8^k) + 2(1 - 8^k) = 8^k + 2$$

b) Linear stability analysis:-

If x^* is an equilibrium point of the dynamical system $x_{k+1} = f(x_k)$

If $|f'(x^*)| < 1$ $\Rightarrow$ x^* is stable equilibrium point

If $|f'(x^*)| > 1$ $\Rightarrow$ x^* is unstable equilibrium point

If $f'(x^*) = 1$ $\Rightarrow$ Test Fails

Example:- Determine the stability of all equilibrium of the dynamical system $x_{k+1} = x_k^3$

Solution:- to determine the equilibrium point $f(x) = x$

$$x^3 - x = 0 \qquad x(x^2 - 1) = 0$$

$x_1^* = 0$ $x_2^* = 1$ $x_3^* = -1$ $\therefore x_1^*,\ x_2^*,\ x_3^*$ are the equilibrium points

$$f'(x) = 3x^2$$

$$f'(x_1^*) = f'(0) = 0 < 1$$

x_1^* is a stable equilibrium point

$$f'(x_2^*) = f'(x_3^*) = f'(\pm 1) = 3 > 1$$

Both x_2^*, x_3^* are unstable equilibrium point

2) Two-Dimensional system:-

Linearization and stability analysis:-

Let λ_1 and λ_2 be the eigenvalue of the Jacobin matrix A.

Real eigenvalue

If $|\lambda_1| < 1$ and , $|\lambda_2| < 1$ $\Rightarrow$ stable node

If $|\lambda_1| > 1$ or $|\lambda_2| > 1$ $\Rightarrow$ unstable node

If $\lambda_i \leq 1, \ |\lambda_j| = 1$ $\Rightarrow$ Test fails

Complex eigenvalue $\lambda = \alpha + i\beta$

If $|\lambda| < 1$ $\Rightarrow$ stable spiral

If $|\lambda| > 1$ $\Rightarrow$ unstable spiral

If $|\lambda| = 1$ $\Rightarrow$ Test fails

Example:- Determine the stability of all equilibrium of the dynamical system $x_{k+1} = f(x_k)$, where $f(x) = \begin{bmatrix} \chi + y^2 \\ x + 2y \end{bmatrix}$

Solution:- to determine the equilibrium point $f(x) = x$

$$\begin{bmatrix} \chi + y^2 \\ x + 2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\chi + y^2 = x \Rightarrow y = 0 \qquad , \qquad x + 2y = y \Rightarrow x = 0$$

$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium point

The jacobian matrix: $A = \begin{bmatrix} 1 & 2y \\ 1 & 2 \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ $\Rightarrow$ the eigenvalues $\lambda_1 = 1, \ \lambda_2 = 2$