2) Two-Dimensional system:-

Linearization and stability analysis:-

Let λ_1 and λ_2 be the eigenvalue of the Jacobin matrix A.

Real eigenvalue

If $|\lambda_1| < 1$ and , $|\lambda_2| < 1$ $\Rightarrow$ stable node

If $|\lambda_1| > 1$ or $|\lambda_2| > 1$ $\Rightarrow$ unstable node

If $\lambda_i \leq 1,\ |\lambda_j| = 1$ $\Rightarrow$ Test fails

Complex eigenvalue $\lambda = \alpha + i\beta$

If $|\lambda| < 1$ $\Rightarrow$ stable spiral

If $|\lambda| > 1$ $\Rightarrow$ unstable spiral

If $|\lambda| = 1$ $\Rightarrow$ Test fails

Example:- Determine the stability of all equilibrium of the dynamical system $x_{k+1} = f(x_k)$, where $f(x) = \begin{bmatrix} \chi + y^2 \\ x + 2y \end{bmatrix}$

Solution:- to determine the equilibrium point $f(x) = x$

$$\begin{bmatrix} \chi + y^2 \\ x + 2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\chi + y^2 = x \Rightarrow y = 0 \qquad , \qquad x + 2y = y \Rightarrow x = 0$$

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is an equilibrium point}$$

The jacobian matrix: $A = \begin{bmatrix} 1 & 2y \\ 1 & 2 \end{bmatrix}$

$$A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow \text{the eigenvalues } \lambda_1 = 1,\ \lambda_2 = 2$$

$$|\lambda_2| > 1$$

The equilibrium point $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is unstable node

III) Lyapunov functions:-

Consider x^* is an equilibrium point of a dynamical system

Let J be a function of x satisfies the following condition:-

I) $v(x)$ is a continuous function

II) $v(x) > 0,\ \forall x \neq x^*$

III) $v(x^*) = 0$

1) Continuous-Time:-

If $\frac{dv}{dt} < 0,\ \forall x$ at least within a fixed positive distance of x^* (x^* may be excluded), then $v(x)$ is a Lyapunov function and x^* is stable

If $\frac{dv}{dt} > 0,\ \forall x \neq x^*$ then x^* is unstable

2) Discrete-Time:-

If $v\big(f(x)\big) - V(x) < 0,\ \forall x$ at least within a fixed positive distance of x^* (x^* may be excluded), then $V(x)$ is a Lyapunov function and x^* is stable

If $Vf\big((x)\big) - v(x) > 0,\ \forall x \neq x^*$ then x^* is unstable

The general form of the function $V(x)$

$$V(x) = (x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 + \cdots + (x_n - x_n^*)^2$$

Example:- consider the dynamical system $x' = -x^3$. Study the equilibrium points and determine their stability

Solution:- $x' = 0 \quad \Rightarrow \quad -x^3 = 0 \quad \Rightarrow \quad x = 0$

One equilibrium point $x^* = 0$

$$f'(x) = -3x^2 \qquad \Rightarrow \qquad f'(0) = 0$$

Linearization fails

Using Lyapunov functions let $V(x) = x^2$ satisfies the conditions:-

I) $V(x)$ is a continuous function

II) $V(x) > 0,\ \forall x \neq x^*$

III) $V(x^*) = 0$

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = 2xx' = (2x)(-x^3) = -2x^4 < 0, \qquad \forall x \neq x^*$$

$x^* = 0$ is a stable equilibrium point

Example:- consider the dynamical system $x' = f(x) = \begin{bmatrix} -y \\ x + y^3 - 3y \end{bmatrix}$. Using Lyapunov functions study the equilibrium points and determine their stability

Solution:- $x' = 0 \quad \Rightarrow \quad -y = 0 \quad \Rightarrow \quad y = 0$

$$x + y^3 - 3y = 0 \qquad \Rightarrow \qquad x = 0$$

The equilibrium points is $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Using Lyapunov functions let $V(x) = x^2 + y^2$ satisfies the conditions:-

I) $V(x)$ is a continuous function

II) $V(x) > 0, \ \forall x \neq x^*$

III) $V(x^*) = 0$

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} + \frac{dV}{dy}\frac{dy}{dt} = 2x(-y) + 2y(x + y^3 - 3y)$$

$$= -2xy + 2yx + 2y^4 - 6y^2$$

$$= 2y^2(y^2 - 3)$$

$\therefore \frac{dV}{dt} < 0 \ \ \forall |y| < \sqrt{3}$, with a circle of radius $\sqrt{3}$ around x^* we have $\frac{dV}{dt} < 0$

$V(x) = x^2 + y^2$ is a Lyapunov function and $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a stable node.

Example:- consider the dynamical system $x_{k+1} = f(x_k), \ x' = f(x) = \begin{bmatrix} y^2 \\ x \end{bmatrix}$. Using Lyapunov functions. Study the equilibrium points and determine their stability

Solution:- $f(x) = x \quad \Rightarrow \quad \begin{bmatrix} y^2 \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \ x = 0, \ \ y = 0$

The equilibrium points is $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Using Lyapunov functions let $V(x) = x^2 + y^2$ satisfies the conditions:-

I) $V(x)$ is a continuous function

II) $V(x) > 0, \ \forall x \neq x^*$

III) $V(x^*) = 0$

$$V\big(f(x)\big) = V(x) = y^2 - x^2 - x^2 - y^2 = y^2(y^2 - 1)$$

$\therefore V\big(f(x)\big) - V(x) < 0,\ \ \forall |y| < 1$, with a circle of radius $\sqrt{3}$ around x^* we have $V\big(f(x)\big) - V(x)$, $V(x) = x^2 + y^2$ is a Lyapunov function and $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a stable node.