# Chapter 2
# Periodicity and chaos

**One-Dimensional discrete time:-**

**1) General form:-**

$$x_{k+1} = f(x_k) \qquad\qquad x(0) = x_0$$

**2) periodic points:-**

Consider the system $x_{k+1} = b - x_k$ , $x(0) = x_0$ where $x_0$ and $b$ are constants

The system can be written $x_0, x_0 - b, \ x_0, \ x_0 - b, \cdots$

$$f^2(x_0) = f^4(x_0) = \cdots = x_0$$

$x_0$ is periodic point of period 2 (the smallest iteration)

Definition 1:- A point $x^*$ is called a periodic point of period P of a dynamical system if P is the smallest number satisfies that $f^P(x^*) = x^*$

Stability the periodic point:-

$|f^{(P)'}(x)| < 1$, $x^*$ is stable periodic point

$|f^{(P)'}(x)| > 1$, $x^*$ is unstable periodic point

$|f^{(P)'}(x)| = 1$, $x^*$ is neutral periodic point

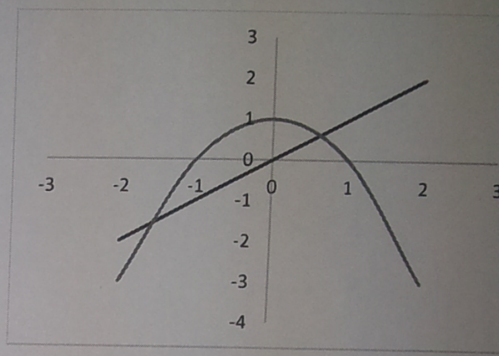
**Example:-** Find the fixed points and the periodic points of period 2 of the system

$$x_{k+1} = 1 - x_k^2$$

**Solution:-** $f(x) = 1 - x^2$

To find the fixed points $f(x) = x$

$x^2 + x - 1 = 0 \quad \Rightarrow \quad x = \frac{-1 \pm \sqrt{5}}{2}$




There are two fixed points $x_1^* = \frac{-1+\sqrt{5}}{2}$, $x_2^* = \frac{-1-\sqrt{5}}{2}$

$f'(x) = -2x \quad \Rightarrow \quad \left|f'\left(\frac{-1\pm\sqrt{5}}{2}\right)\right| > 1$

Both $\frac{-1\pm\sqrt{5}}{2}$ are unstable fixed point

to determine the periodic points of period 2, let $f^2(x) = x$

$$f^2(x) = f(1 - x^2) = 1 - (1 - x^2)^2 = 1 - 1 + 2x^2 - x^4 = 2x^2 - x^4$$

$$\therefore 2x^2 - x^4 = x \quad \Rightarrow \quad x^4 - 2x^2 + x = 0$$

$$x(x^3 - 2x + 1) = 0 \Rightarrow x(x-1)\underbrace{(x^2+x-1)}=0$$

$\Downarrow$

$x = \frac{-1 \pm \sqrt{5}}{2} \rightarrow$ fixed point

Periodic points of period 2: $x = 0 \quad x = 1$

$$(f^2(x))' = -4x^3 + 4x$$

$|f^{(2)\prime}(0)| = 0 < 1 \quad \Rightarrow \quad x = 0$ is a stable periodic point of period 2

$|f^{(2)\prime}(1)| = 0 < 1 \quad \Rightarrow \quad x = 1$ is a stable periodic point of period 2

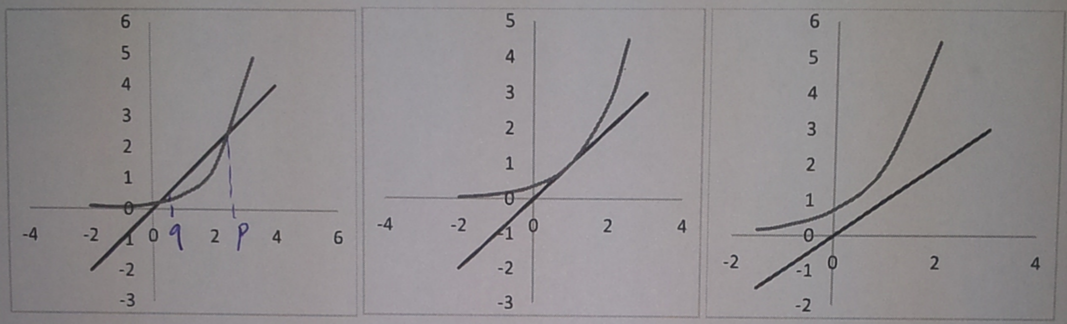
**3) Bifurcation:-**

Bifurcation theory studies the changes that's maps undergo as some parameter changes. For the change on periodic points structure, there are two main types of Bifurcation.

## I) Tangent bifurcation:-

As a parameter changes, the fixed point structure changes from no fixed points to a unique fixed point~~s~~, then a pair of fixed points (stable and unstable) are obtained.

For example:- the family $E_\lambda = \lambda e^x, \ \lambda > 0$




$0 < \lambda < \frac{1}{e}$ Two fixed points p is unstable q is stable

$\lambda = \frac{1}{e}$ One fixed point $x = 1$

$\lambda > \frac{1}{e}$ NO fixed point

## II) Period-Doubling bifurcation:-

As a parameter changes, a fixed point changes from attracting (stable) to neutral, then to repelling (unstable)

For example:- The family $f_\mu(x) = \mu x(1-x)$ $\quad \therefore x \overset{=0}{} $ is a fixed point

The $f'_\mu(x) = \mu - 2\mu x \quad \Rightarrow \quad |f'_\mu(0)| = \mu$

If $\mu < 1 \quad \Rightarrow \quad x = 0$ is stable

If $\mu = 1 \quad \Rightarrow \quad x = 0$ is neutral

If $\mu > 1 \quad \Rightarrow \quad x = 0$ is unstable

As the parameter $\mu$ increased through 1, the fixed point $x = 0$ changes from stable to neutral to unstable

The family $F_\mu(x)$ undergoes period-doubling bifunction

**Example:-** Discuss the bifurcation which occurs in the family $f_\mu(x) = \mu x(1-x)$, $\mu > 1$

**Solution:-** to determine the fixed point let $f_\mu(x) = x$

$$\mu x(1-x) = x$$

$$x(\mu(1-x)-1) = 0 \quad \Rightarrow \quad x(\mu - \mu x - 1) = 0$$

$$x = 0 \quad \text{or} \quad \mu - \mu x - 1 = 0 \quad \Rightarrow \quad x = \frac{\mu-1}{\mu}$$

$|f'_\mu(0)| = \mu > 1$ $\qquad$ $x = 0$ is unstable fixed point

$$\left|f'_\mu\left(\frac{\mu-1}{\mu}\right)\right| = |\mu - 2(\mu-1)| = |2-\mu|$$

If $1 < \mu < 3$, $\left|f'_\mu\left(\frac{\mu-1}{\mu}\right)\right| < 1 \quad \Rightarrow \quad x = \frac{\mu-1}{\mu}$ is stable fixed point

If $\mu = 3$, $\left|f'_\mu\left(\frac{\mu-1}{\mu}\right)\right| = 1 \quad \Rightarrow \quad x = \frac{\mu-1}{\mu}$ is ~~stable~~ fixed neutral point

If $\mu > 3$, $\left|f'_\mu\left(\frac{\mu-1}{\mu}\right)\right| > 1 \quad \Rightarrow \quad x = \frac{\mu-1}{\mu}$ is unstable fixed point

The family $f_\mu(x) = \mu x(1-x)$ undergoes a period-doubling bifurcation through $\mu = 3$.

**Example:-** Discuss the bifurcation which occurs in the family $Q_c(x) = x^2 + c$

III Transcritical bifurcation (page 135).