## 4) Chaotic Dynamical System

Definition 2:- For any sets, the closure of $S(S)$ contains all points in $S$ and all the limit points of $S$.

For example:- $S = (0,1)$, the $S = [0,1]$ if $S$ is closed set, the $S = S$

Definition 3:- A subset $U$ of $S$ is dense in $S$ if $\overline{U} = S$

For example:- the set of rational numbers is dense in the real number.

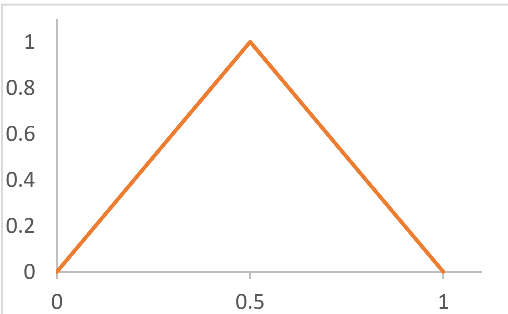
Every open set of $\mathbb{R}$ contains a rational number

**Example:-** consider the test function $T: [0,1] \to [0,1]$, such that

$$T(x) = \begin{cases} 2x & , x \leq \frac{1}{2} \\ 2 - 2x & , x > \frac{1}{2} \end{cases}$$

**Solution:-**

$$T^2(x) = \begin{cases} 4x & , x \leq \frac{1}{4} \\ 2 - 4x & , \frac{1}{4} \leq x \leq \frac{1}{2} \\ 4x - 2 & , \frac{1}{2} \leq x \leq \frac{3}{4} \\ 4 - 4x & , \frac{3}{4} \leq x \leq 1 \end{cases}$$




$$T^2(x) = T(T(x))$$

Definition 4:- A map $f: J \to J$ has sensitive dependence on initial conditions if there is $\delta > 0$ such that for any $x \in J$, and any neighborhood $N$ of $x$, there exist $y \in N$ and $n \geq 0$, such that $\left|f^{(n)}(x) - f^{(n)}(y)\right| > \delta$.

Definition 5:- A map $f: J \rightarrow J$ is said to be topologically transitive if any pair of open sets $U, v \subset J$ there exists $k > 0$, such that $f^k(U) \cap v \neq \emptyset$.

Topologically transitive systems does not decomposed

Definition 6:- A map $f: V \rightarrow V$ is said to be chaotic on $V$ if

I) $f$ has sensitive dependence on initial conditions

II) $f$ is topologically transitive

III) periodic points are dense in $V$ (every open set in $V$ contains a periodic points)

The main features of chaotic dynamical system:-

I) unpordictability

II) in decomposability

III) element of regularity

**Example:-** let $s^1$ denote the unit circle of the plane. Prove that the map $f: s^1 \rightarrow s^1$

Given by $f(\theta) = 2\theta$ is chaotic

**Solution:-**

1) The angular distance between any two points is doubled under the iteration of $f$, then $f$ has sensitive dependence on initial condition

2) $f$ is topologically transitive, because any orbit in $s^1$ is expanded by some $f^k$ to cover all of $s^1$, and interest with any other orbit is $s^1$

3) to determine the periodic points put $f^P(\theta) = \theta + 2n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$

$$2^p\theta = \theta + 2n\pi \quad \Rightarrow \qquad \theta(2^p - 1) = 2n\pi$$

$$\theta = \frac{2n\pi}{2^p - 1}$$

The set of periodic points is dense in $s^1$.

The map $f: s^1 \to s^1 : f(\theta) = 2\theta$ is chaotic

**5) Lyapunov Exponents:-**

Definition:- consider two nearby points $x_0$ and $x_0 + \delta_0$, where $\delta_0$ is small. If the separation after iterations, $\delta_n$ is given by $|\delta_n| \approx |\delta_0|e^{n\lambda}$, then $\lambda$ is called Lyapunov exponents.

$$\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$$

Theorem 1:- A positive Lyapunov exponent means chaos

Theorem 2:- The Lyapunov exponent

$$\lambda = \lim_{n\to\infty}[\frac{1}{n}\sum_{i=0}^{n-1} ln|f'(x_i)|]$$

Proof:- From the definition of Lyapunov exponents $|\delta_n| \approx |\delta_0|e^{n\lambda}$

$$\lambda = \frac{1}{n}\ln\left(\frac{|\delta_n|}{|\delta_0|}\right) = \frac{1}{n}\ln\left|\frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0}\right|$$

$$\therefore \lambda = \frac{1}{n}\ln|(f^n)'(x_0)|$$

$$\frac{d}{dx}(f^2(x))_{x=x_0} = \frac{d}{dx}\Big(f\big(f(x)\big)\Big)_{x=x_0} = f'\big(f(x_0)\big)f'(x_0) = f'(x_1)f'(x_0)$$

$$\frac{d}{dx}(f^3(x))_{x=x_0} = f'(x_2)f'(x_1)f'(x_0)$$

$$\therefore \frac{1}{n}\ln|(f^n)'(x_0)| = \frac{1}{n}\ln\left|\prod_{i=0}^{n-1} f'(x_i)\right| = \frac{1}{n}\sum_{i=0}^{n-1} ln|f'(x_i)|$$

As $n \to \infty$

$$\lambda = \lim_{n\to\infty}[\frac{1}{n}\sum_{i=0}^{n-1} ln|f'(x_i)|]$$

**Example:-** study the dynamics of the map $f: s^1 \to s^1 : f(\theta) = 2\theta$

**Solution:-** $f'(\theta) = 2$

The Lyapunov exponents

$$\lambda = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=0}^{n-1} ln|f'(\theta)|\right] = \ln 2 > 0$$

The map $f(\theta) = 2\theta$ is chaotic

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**Example:-** study and describe the dynamics of the map

$$f(x) = \begin{cases} rx & ,0 \leq x \leq \frac{1}{2} \\ r - rx & ,\frac{1}{2} \leq x \leq 1 \end{cases}$$

**Solution:-** $f'(x) = \pm r$

$$\lambda = \ln|r|$$

$|r| > 1, \lambda > 0 \quad \Rightarrow \quad f(x)$ is chaotic

$|r| \leq 1, \lambda \leq 0 \quad \Rightarrow \quad f(x)$ is not chaotic