4. LAURENT SERIES

Theorem VII. Laurent's theorem [equation (4.1)] (which we shall state without proof). Let C_1 and C_2 be two circles with center at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$(4.1)\quad f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

convergent in R.

Such a series is called a *Laurent series*. The "b" series in (4.1) is called the *principal part* of the Laurent series.

The formulas for the coefficients in (4.1) are (Problem 5.2)

$$(4.3)\qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \qquad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}},$$

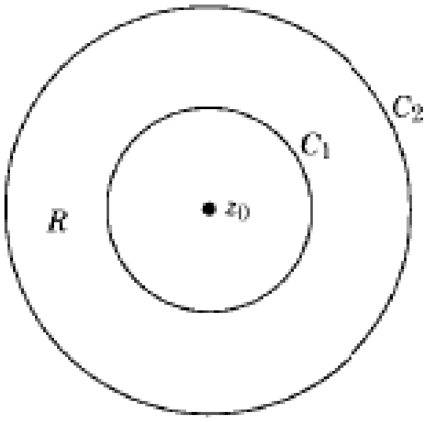

FIGURE 4.1

Definitions:

If all the b's are zero, $f(z)$ is analytic at $z = z_0$, and we call z_0 a *regular point*. (See Problem 4.1.)

If $b_n \neq 0$, but all the b's after b_n are zero, $f(z)$ is said to have a *pole of order* n at $z = z_0$. If $n = 1$, we say that $f(z)$ has a *simple pole*.

If there are an infinite number of b's different from zero, $f(z)$ has an *essential singularity* at $z = z_0$.

The coefficient b_1 of $1/(z - z_0)$ is called the *residue* of $f(z)$ at $z = z_0$.

14. Obtain all the Laurant series of the function $\frac{7z-2}{(z+1)z(z-2)}$ about $z = -1$.

(JNTU 2006 Aug., 2007 Feb., 2008 April/May)

Solution: $f(z) = \frac{7z-2}{(z+1)z(z-2)}$

Put

$$z + 1 = u$$

$$z = u - 1$$

$$z - 2 = u - 3$$

$$\frac{7z-2}{(z+1)z(z-2)} = \frac{7(u-1)-2}{u(u-1)(u-3)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u-3}$$

$$A = \lim_{u\to 0} \frac{7u-9}{(u-1)(u-3)} = -3$$

$$B = \lim_{u\to 1} \frac{7u-9}{u(u-3)} = 1$$

$$C = \lim_{u\to 3} \frac{7u-9}{u(u-1)} = 2$$

$$\therefore\ -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3} = -\frac{3}{u} - (1-u)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u} - (1 + u + u^2 + u^3 + \cdots) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{9} + \cdots\right)$$

$$= -\frac{3}{u} - \frac{5}{3} - \left(1+\frac{2}{3^2}\right)u - \left(1+\frac{2}{3^3}\right)u^2 - \cdots$$

$$= -\frac{3}{z+1} - \frac{5}{3} - \left(1+\frac{2}{3^2}\right)(z+1) - \left(1+\frac{2}{3^3}\right)(z+1)^2 - \left(1+\frac{2}{3^4}\right)(z+1)^3 + \cdots$$

15. Expand $\dfrac{1}{(z^2-3z+2)}$ is the region

(i) $0 < | z - 1 | < 1$ (ii) $1 < | z | < 2$

(iii) $| z | > 2$.

(JNTU 2004 Nov., 2006 April, 2006 Oct., 2007 Feb., 2008, 2009 Nov., 2010 June)

Solution:

(i)
$$\frac{1}{(z^2-3z+2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$$
$$| z - 1 | < 1$$
$$\frac{1}{(z-2)}-\frac{1}{(z-1)}=\frac{1}{(z-1-1)}-\frac{1}{(z-1)}$$
$$=-\frac{1}{[1-(z-1)]}-\frac{1}{(z-1)}$$
$$=-[1-(z-1)]^{-1}-\frac{1}{(z-1)}$$
$$=-(1+(z-1)+(z-1)^2+(z-1)^3+\cdots)-\frac{1}{(z-1)}$$

(ii) $1 < | z |, | z | > 1, \dfrac{1}{|z|} < 1, \dfrac{|z|}{2} < 1$

$$\frac{1}{(z-2)}-\frac{1}{(z-1)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$
$$=-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

(iii) $\dfrac{2}{|z|}<1.$

$$\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}$$
$$=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$
$$=\frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$
$$=\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n}-\sum_{n=1}^{\infty}\frac{1}{z^n}$$
$$=\sum_{n=1}^{\infty}\frac{1}{z^n}(2^{n-1}-1).$$

Find Laurent series about the indicated singularity for each of the following functions:

(a) $\dfrac{e^{2z}}{(z-1)^3}$; $z = 1$. (c) $\dfrac{z - \sin z}{z^3}$; $z = 0$. (e) $\dfrac{1}{z^2(z-3)^2}$; $z = 3$.

(b) $(z-3)\sin\dfrac{1}{z+2}$; $z = -2$. (d) $\dfrac{z}{(z+1)(z+2)}$; $z = -2$.

Name the singularity in each case and give the region of convergence of each series.

Solution

(a) Let $z - 1 = u$. Then $z = 1 + u$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3}\cdot e^{2u} = \frac{e^2}{u^3}\left\{1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \frac{(2u)^4}{4!} + \cdots\right\}$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

$z = 1$ is a *pole of order* 3, or *triple pole*.

The series converges for all values of $z \neq 1$.

(b) Let $z + 2 = u$ or $z = u - 2$. Then

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u} = (u-5)\left\{\frac{1}{u} - \frac{1}{3!\,u^3} + \frac{1}{5!\,u^5} - \cdots\right\}$$

$$= 1 - \frac{5}{u} - \frac{1}{3!\,u^2} + \frac{5}{3!\,u^3} + \frac{1}{5!\,u^4} - \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$$

$z = -2$ is an *essential singularity*.

The series converges for all values of $z \neq -2$.

(c)

$$\frac{z - \sin z}{z^3} = \frac{1}{z^3}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right\}$$

$$= \frac{1}{z^3}\left\{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right\} = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$$

$z = 0$ is a *removable singularity.*

The series converges for all values of z.

(d) Let $z + 2 = u$. Then

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)u} = \frac{2-u}{u} \cdot \frac{1}{1-u} = \frac{2-u}{u}(1 + u + u^2 + u^3 + \cdots)$$

$$= \frac{2}{u} + 1 + u + u^2 + \cdots = \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$$

$z = -2$ is a *pole of order* 1, or *simple pole.*

The series converges for all values of z such that $0 < |z+2| < 1$.

(e) Let $z - 3 = u$. Then, by the binomial theorem,

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(3+u)^2} = \frac{1}{9u^2(1+u/3)^2}$$

$$= \frac{1}{9u^2}\left\{1 + (-2)\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \cdots\right\}$$

$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots$$

$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \cdots$$

$z = 3$ is a *pole of order* 2 or *double pole.*

The series converges for all values of z such that $0 < |z-3| < 3$.

Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for:

(a) $1 < |z| < 3$, (b) $|z| > 3$, (c) $0 < |z+1| < 2$, (d) $|z| < 1$.

Solution

(a) Resolving into partial fractions,

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

If $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then, the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$, i.e., $1 < |z| < 3$, is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(b) If $|z| > 1$, we have as in part (a),

$$\frac{1}{2(z+1)} = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| > 3$,

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e., $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(c) Let $z + 1 = u$. Then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$
$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

valid for $|u| < 2$, $u \neq 0$ or $0 < |z+1| < 2$.

(d) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (a),

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion, valid for both $|z| < 1$ and $|z| < 3$, i.e., $|z| < 1$, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

This is a *Taylor series*.