Lecture # 8
Combinatorics
Stirling Numbers II, I

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Stirling numbers in terms of factorial powers.

Stirling numbers are often defined as the coefficients in an expansion of positive integral powers of a variable in terms of factorial powers, or vice-versa:

Notation and Standard Formulae

Def (1). (Falling factorial)

The n^{th} **falling factorial** of real number x of order n; n is positive integer, denoted by $(x)_n$, $x^{(n)}$ or $x^{\underline{n}}$, is defined as the product of n factors:

$$(x)_n := x(x-1)(x-2)...(x-n+1) = n!\binom{x}{n} = \prod_{i=0}^{n-1}(x-i), \text{ for } n = 1,2,... \quad (1)$$

and $(x)_0 = 1$.

We deduce the following fundamental property of the factorials in the following lemma:

Lemma (5). If r is positive integer, then

$$(x)_{k+r} = (x)_k (x-k)_r. \quad (2)$$

Proof. If r is positive integer, then

$$(x)_{k+r} := x(x-1)(x-2)...(x-k-r+1)$$

$$=\left[x(x-1)...(x-k+1)\right]\left[(x-k)(x-k-1)...(x-k-r+1)\right].$$

Since $(x-k)(x-k-1)...(x-k-r+1)=(x-k)_r$, then

$$=\left[x(x-1)...(x-k+1)\right](x-k)_r=(x)_k(x-k)_r$$, i.e.,

$$(x)_{k+r}=\frac{x!}{(x-k-r)!}=\frac{x!}{(k-r)!}\frac{(k-r)!}{(x-k-r)!}=(x)_k(x-k)_r.$$

We have from this formula, for $r>0$ and $k=0$, that

$$(x)_{0+r}=(x)_0(x-0)_r=1(x)_r.$$ Thus, $(x)_0=1$.

<u>**Lemma (6)**</u> (Vandermonde Formula)

Let x and y be real numbers and n a positive integer. Then

$$(x+y)_n=\sum_{k=0}^{n}\binom{n}{k}(x)_k(y)_{n-k}. \qquad (3)$$

Proof. (By Induction)

It's easy to show formula (3) is true for the two cases $n=1,2$.
Assume that it's true for $n-1$, that is

$$(x+y)_{n-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_k(y)_{n-k-1}.$$

It will be shown that it holds also for n, we have

$$(x+y)_n=(x+y-n+1)(x+y)_{n-1}$$

$$=(x+y-n+1)\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_k(y)_{n-k-1}.$$

Since $(x+y-n+1)(x)_k(y)_{n-k-1}=(x)_k(y)_{n-k}+(x)_{k+1}(y)_{n-k-1}$,

we successively get

$$(x+y)_n=\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_k(y)_{n-k}+\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_{k+1}(y)_{n-k-1}$$

$$=\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_k(y)_{n-k}+\sum_{k=1}^{n-1}\binom{n-1}{k-1}(x)_k(y)_{n-k}$$

$$=(y)_n+\sum_{k=1}^{n-1}\left\{\binom{n-1}{k}+\binom{n-1}{k-1}\right\}(x)_k(y)_{n-k}+(x)_k .$$

Therefore, on using **Pascal's triangle**, we deduce the result.

Lemma (7). The defining equation for **Stirling** numbers of the **second** kind, using formula (1), can be written as

$$x^n=\sum_{j=0}^{n}S(n,j)(x)_j\equiv\sum_{j=0}^{n}\begin{Bmatrix}n\\ j\end{Bmatrix}(x)_j,\ \text{for } n=1,2,\ldots \qquad (4)$$

Proof. (By Induction on n),

For $n=0$ gives $1=1$, the formula is true, so assume it’s true for $n-1$.

Note that $(x)_{k+1}=(x)_k(x-k)$. So, $x(x)_k=(x)_{k+1}+k(x)_k$. We have

$$x^n=xx^{n-1}=x\left(\sum_{k=0}^{n-1}\begin{Bmatrix}n-1\\ k\end{Bmatrix}(x)_k\right)=\sum_{k=0}^{n-1}\begin{Bmatrix}n-1\\ k\end{Bmatrix}\left((x)_{k+1}+k(x)_k\right)$$

$$= \sum_{k=0}^{n-1} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} (x)_{k+1} + \sum_{k=0}^{n-1} k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} (x)_k = \sum_{k=0}^{n-1} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} (x)_k + \sum_{k=0}^{n-1} k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} (x)_k$$

$$= \sum_{k=0}^{n-1} \left(\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} \right) (x)_k = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (x)_k .$$

The result follows.

If we let $(x)_n := x(x-1)(x-2)...(x-n+1)$ be the falling factorial, we can characterize the Stirling numbers of the second kind by (4) (the notation that combinatorialists use for falling factorials coincides with the notation used in special functions for rising factorials). Note that formula (4) is precisely the recurrence obeyed by $S(n, j)$,

$$S(n+1,k) = k\, S(n,k) + S(n,k-1),$$

if the conditions $S(n,n)=1,\ S(n,k)=0,$ for $k \le 0$ and $k \ge n+1$, are used: Now

$$x^{n+1} = \sum_{k=-\infty}^{\infty} S(n+1,k)(x)_k , \qquad (5)$$

$$x^{n+1} = x x^n. \qquad (6)$$

Substitution of equations (4) and (5) into equation (5) gives

$$\sum_{k=-\infty}^{\infty} S(n+1,k)(x)_k = x \sum_{k=-\infty}^{\infty} S(n,k)(x)_k = \sum_{k=-\infty}^{\infty} S(n,k)\, x(x)_k . \ (7)$$

However, $x(x)_k = xx(x-1)...(x-k+1)$

$$= (x-k+k)x(x-1)...(x-k+1) = (x)_{k+1} + k(x)_k .$$

Using this last result in equation (7), we obtain

$$\sum_{k=-\infty}^{\infty} S(n+1,k)(x)_k = \sum_{k=-\infty}^{\infty} S(n,k)\left((x)_{k+1} + k(x)_k\right)$$

$$= \sum_{k=-\infty}^{\infty} S(n,k)(x)_{k+1} + \sum_{k=-\infty}^{\infty} kS(n,k)(x)_k$$

$$= \sum_{k=-\infty}^{\infty} S(n,k-1)(x)_k + \sum_{k=-\infty}^{\infty} kS(n,k)(x)_k ,$$

Comparing the coefficient of $(x)_n$ implies the result.

Stirling numbers of the second kind are the coefficients in the factorial polynomials

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k = S(n,0)(x)_0 + S(n,1)(x)_1 + \ldots + S(n,n)(x)_n$$

$$= \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (x)_k = \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} (x)_0 + \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} (x)_1 + \ldots + \left\{ \begin{matrix} n \\ n \end{matrix} \right\} (x)_n .$$

Lemma. The Stirling numbers of the second kind, $S(n,k)$ satisfy the following formulae, for any two positive integers $n,k,\ n \ge k$:

$$S(n,k) = \sum_{u=k-1}^{n-1} \binom{n-1}{u} S(u,k-1), \text{ and}$$

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n .$$

We find on setting $n = 1,2,3,\ldots,$ and expanding $(x)_n$ we have

$$n=1:\ (x)_1 = x \qquad\qquad = 1x,$$
$$n=2:\ (x)_2 = x(x-1) \qquad\qquad = -1x+1x^2,$$
$$n=3:\ (x)_3 = x(x-1)(x-2) \qquad\qquad = 2x-3x^2+1x^3, \qquad \text{(A)}$$
$$n=4:\ (x)_4 = x(x-1)(x-2)(x-3) \qquad\qquad = -6x+11x^2-6x^3+1x^4,$$
$$n=5:\ (x)_5 = x(x-1)(x-2)(x-3)(x-4) = 24x-50x^2+35x^3-10x^4+1x^5,$$

etc.

Arranging the coefficients in a triangle array gives the **Stirling triangle** of numbers of the **first kind**.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & -1 & & 1 & & & \\ & & 2 & & -3 & & 1 & & \\ & -6 & & 11 & & -6 & & 1 & \\ 24 & & -50 & & 35 & & -10 & & 1 \end{array}$$

Stirling triangle of the first kind

These relations, (A) , can be inverted to give the various powers of x in terms of the factorial functions $(x)_n$:

$$x = 1(x)_1,$$
$$x^2 = 1(x)_1 + 1(x)_2,$$
$$x^3 = 1(x)_1 + 3(x)_2 + 1(x)_3,$$
$$x^4 = 1(x)_1 + 7(x)_2 + 6(x)_3 + 1(x)_4, \qquad \text{(B)}$$
$$x^5 = 1(x)_1 + 15(x)_2 + 25(x)_3 + 10(x)_4 + 1(x)_5,$$
$$x^6 = 1(x)_1 + 31(x)_2 + 90(x)_3 + 65(x)_4 + 15(x)_5 + 1(x)_6,$$

etc.

Arranging the coefficients in a triangle array gives

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 3 & & 1 & & & \\
 & & 1 & & 7 & & 6 & & 1 & & \\
 & 1 & & 15 & & 25 & & 10 & & 1 & \\
1 & & 31 & & 90 & & 65 & & 15 & & 1
\end{array}
$$

Stirling triangle of the second kind

For example, $x^3 = \left\{\begin{matrix}3\\0\end{matrix}\right\}(x)_0 + \left\{\begin{matrix}3\\1\end{matrix}\right\}(x)_1 + \left\{\begin{matrix}3\\2\end{matrix}\right\}(x)_2 + \left\{\begin{matrix}3\\3\end{matrix}\right\}(x)_3$

$$= \left\{\begin{matrix}3\\0\end{matrix}\right\} + \left\{\begin{matrix}3\\1\end{matrix}\right\}x + \left\{\begin{matrix}3\\2\end{matrix}\right\}x(x-1) + \left\{\begin{matrix}3\\3\end{matrix}\right\}x(x-1)(x-2)$$

$$= 0 + x + 3\left(x^2 - x\right) + \left(x^3 - 3x^2 + 2x\right).$$

The expressions given in equations (A) and (B) can be written in the following generated forms, respectively:

$$(x)_n = \sum_{k=0}^{n} s(n,k)\,x^k,\ n \ge 1 \qquad \text{(A)}_1$$

and Stirling numbers of the second kind are the coefficients in the factorial polynomials

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k,\ n \ge 1\ . \qquad \text{(B)}_1$$

The coefficients $s(n,k)$ are called **Stirling** numbers of the **first kind**, while the coefficients $S(n,k)$ are called **Stirling** numbers of the **second kind**.

These numbers satisfy the following recurrence relations:

$$s(n+1,k)=s(n,k-1)-ns(n,k), \qquad (A)_2$$

where for $n>0$, $s(n,n)=1$, $s(n,k)=0$, for $k\le 0$ and $k\ge n+1$, and

$$S(n+1,k)=S(n,k-1)-kS(n,k), \qquad (B)_2$$

where for $n>0$, $S(n,n)=1$, $S(n,k)=0$, for $k\le 0$ and $k\ge n+1$.

Proof. Using the fact $s(n,n)=1$, $s(n,k)=0$, for $k\le 0$ and $k\ge n+1$, where for $n>0$, we can rewrite (A)$_1$ as follows:

$$(x)_n=\sum_{k=-\infty}^{\infty} s(n,k)x^k . \qquad \text{(i)}$$

Therefore

$$(x)_{n+1}=\sum_{k=-\infty}^{\infty} s(n+1,k)x^k . \qquad \text{(ii)}$$

From the definition of the factorial polynomials, we have

$$(x)_{n+1}=(x-n)(x)_n . \qquad \text{(iii)}$$

Substituting equations (i) and (ii) into equation (iii) gives

$$\sum_{k=-\infty}^{\infty} s(n+1,k)x^k=(x-n)\sum_{k=-\infty}^{\infty} s(n,k)x^k$$

$$=\sum_{k=-\infty}^{\infty} s(n,k)x^{k+1}-\sum_{k=-\infty}^{\infty} ns(n,k)x^k$$

$$=\sum_{k=-\infty}^{\infty} s(n,k-1)x^k-\sum_{k=-\infty}^{\infty} ns(n,k)x^k .$$

If we equate the coefficients of x^k, we obtain the result (A)$_2$.

The defining equation for **Stirling** numbers of the **second** kind can be written as

$$x^n = \sum_{k=-\infty}^{\infty} S(n,k)(x)_k . \qquad \text{(iv)}$$

If the conditions $S(n,n)=1,\ S(n,k)=0,$ for $k \le 0$ and $k \ge n+1$, are used. Now

$$x^{n+1} = \sum_{k=-\infty}^{\infty} S(n+1,k)(x)_k , \qquad \text{(v)}$$

and

$$x^{n+1} = xx^n . \qquad \text{(vi)}$$

Substitution of equations (iv) and (v) into equation (vi) gives

$$\sum_{k=-\infty}^{\infty} S(n+1,k)(x)_k = x\sum_{k=-\infty}^{\infty} S(n,k)(x)_k = \sum_{k=-\infty}^{\infty} S(n,k)x(x)_k . \ \text{(vii)}$$

However, $x(x)_k = xx(x-1)...(x-k+1)$

$$= (x-k+k)x(x-1)...(x-k+1) = (x)_{k+1} + k(x)_k .$$

Using this last result in equation (vii), we obtain

$$\sum_{k=-\infty}^{\infty} S(n+1,k)(x)_k = \sum_{k=-\infty}^{\infty} S(n,k)\left((x)_{k+1} + k(x)_k\right)$$

$$= \sum_{k=-\infty}^{\infty} S(n,k)(x)_{k+1} + \sum_{k=-\infty}^{\infty} kS(n,k)(x)_k$$

$$= \sum_{k=-\infty}^{\infty} S(n,k-1)(x)_k + \sum_{k=-\infty}^{\infty} kS(n,k)(x)_k ,$$

and the result follows.

Exercises:

1-Write a computer program to evaluate Stirling numbers by means of the recurrence relation:

$$s(n+1,k) = s(n,k-1) - ns(n,k). \qquad \text{(a)}$$

2- Polynomials in x can be expressed in terms of $(x)_1, (x)_2, (x)_3, \ldots$ provided $x, x^2, x^3, \ldots$ have been expressed in this form. Hence, let $x^n = \sum_{k=0}^{n} t(n,k)(x)_k$.

i- Derive a recurrence relation for $t(n,k)$ analogous to formula (a) in exercise (1).

ii- Construct a table of values of $t(n,k)$.

iii- Use the table constructed in (ii) to express the polynomial $x^4 - 4x^3 + 6x^2 - 3x$ in terms of factorials.

"Ans. $x(x-1)^2 + x(x-1)(x-2)^2$

".