Third Year Stats. & Comp.
Stochastic Process
Date: Saturday 5-4-2020
Time: Two hours

Lecture # 8
First passage
(Continuation)

Faculty of Science
Department of Mathematics

Preliminary Facts

Let S_1 be the set of transient states. We will often use conditional probabilities and expectations given $X_0 = i$, and express them as

$$P_i(A) = \Pr(A | X_0 = i), \qquad E_i[X] = E[X | X_0 = i].$$

For instance, $P_i(X_n = j) = p_{ij}^{(n)}$.

From the first entrance theorem (Lecture 7):

For every two states $i, j \in SS$ in a MC $\{X_n, n = 0,1,2,\ldots\}$, the relation of probability $p_{ij}^{(n)}$ in terms of $f_{ij}^{(n)}$ is given by

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}, \text{ for } n = 1,2,\ldots.$$

This expression can also be written in the following form:

$$p_{ij}^{(n)} = \sum_{k=1}^{n-1} f_{ij}^{(k)} p_{jj}^{(n-k)} + f_{ij}^{(n)} \text{ for } n = 2,3,\ldots,$$

where the last equation follows from the fact that $p_{jj}^{(0)} = 1$. From this we have with, $f_{ij}^{(0)} = 0$, that

$$f_{ij}^{(n)} = \begin{cases} p_{ij}^{(n)}, & n = 1 \\ p_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} p_{jj}^{(n-k)}, & n = 2,3,\ldots \end{cases}.$$

Corollary (1). A general recursive for computing $f_{ij}^{(n)}$, for $i, j \in SS$ is $f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} p_{jj}^{(n-k)}$, for $n \geq 2$.

Prof. M A El-Shehawey

Proof. The first passage time probabilities satisfy a recursive relationship:

$$p_{ij}^{(1)}\left(=p_{ij}\right)=f_{ij}^{(1)} \quad\Rightarrow\quad f_{ij}^{(1)}=p_{ij}^{(1)}\left(=p_{ij}\right),$$
$$p_{ij}^{(2)}=f_{ij}^{(2)}+f_{ij}^{(1)}p_{jj}^{(1)} \quad\Rightarrow\quad f_{ij}^{(2)}=p_{ij}^{(2)}-f_{ij}^{(1)}p_{jj},$$
$$p_{ij}^{(3)}=f_{ij}^{(3)}+f_{ij}^{(1)}p_{jj}^{(2)}+f_{ij}^{(2)}p_{jj}^{(1)} \quad\Rightarrow\quad f_{ij}^{(3)}=p_{ij}^{(3)}-f_{ij}^{(1)}p_{jj}^{(2)}+f_{ij}^{(2)}p_{jj}.$$

In general, since

$$p_{ij}^{(n)}=\sum_{r=1}^{n}f_{ij}^{(r)}p_{jj}^{(n-r)}=f_{ij}^{(n)}+\sum_{r=1}^{n-1}f_{ij}^{(r)}p_{jj}^{(n-r)},\quad n\geq 2$$
$$\Rightarrow f_{ij}^{(n)}=p_{ij}^{(n)}-\sum_{r=1}^{n-1}f_{ij}^{(r)}p_{jj}^{(n-r)},\quad n\geq 2..$$

Theorem (1).

Let $\{X_n, n=0,1,...\}$ be a MC with TPM $\mathbf{M}$ over a state space SS. Then for any state $j\in SS$, the following are equivalent:

i- j is recurrent.
ii- $P_j\left(T_j<\infty\right)=1$.
iii- $\sum_{n=0}^{\infty}p_{jj}^{(n)}=\infty$.

Proof. $(i-\Leftrightarrow ii-)$. Suppose that j is **recurrent**. By definition, the MC must hit state j infinitely often. In particular, the MC must hit the state j at least twice. This guarantees the existence of non-negative integers $m<n$ such that:

$$X_m=j \text{ and } X_n=j.$$

By the Markov property and our assumption that the MC is time-homogeneous,

$$\Pr\left(X_n = j, X_m = j\right) = \Pr\left(X_{n-m} = j, X_0 = j\right).$$

Which holds for all $m < n$. Therefore,

$$P_j\left(T_j < \infty\right) = P_j\left(n - m < \infty\right) = 1.$$

Conversely, suppose that T_j is always finite. Then, suppose by contradiction that j is a transient state.

$$\exists\ n > 0 \text{ such that } X_n = j \text{ and } \forall\ m > n,\ X_m \neq j.$$

Then by the Markov Property,

$$\Pr\left\{X_n = j, X_m \neq j\,\forall m < n\right\} = \Pr\left\{X_0 = j, X_n \neq j\,\forall n > 0\right\}.$$

However, if this occurs, it must be true that:

$$T_j \equiv T_{jj} = \min\left\{n \geq 1,\ X_n = j \middle| X_0 = j\right\} \equiv \min\left\{n \geq 1,\ X_n = j\right\} = \infty.$$

But this contradicts our assumption that the return time is finite. $(ii- \Leftrightarrow iii-)$. We define the number of visits to j for a MC starting at j to be: $\nu_j = \sum_{n=0}^{\infty} 1_{\{X_n = j, X_0 = j\}}$. Then we consider:

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \sum_{n=0}^{\infty} \Pr\left(X_n = j, X_0 = j\right) = \sum_{n=0}^{\infty} E\left[1_{\{X_n = j, X_0 = j\}}\right]$$

$$= E\left[\sum_{n=0}^{\infty} 1_{\{X_n = j, X_0 = j\}}\right] = E\left[\nu_j\right].$$

But ν_j is infinite if and only if T_j is finite.

The next lemma demonstrates that recurrence is communication class property:

<u>**Lemma** (1)</u>. Suppose $\{X_n, n = 0,1,...\}$ is a MC with state space SS, suppose C is a communication class. Then given any state j in C:

$$j \in C \text{ is } \textbf{transient} \Leftrightarrow k \in C \text{ is } \textbf{transient } \forall\, k \in C$$

Proof. Suppose that $j \in C$ is **transient** and fix $k \in SS$. Since C is a communication class, there exist positive integers m, n such that: $p_{jk}^{(m)} > 0$ and $p_{kj}^{(n)} > 0$. Then we observe that for all $r \geq 0$:

$$p_{jj}^{(m+r+n)} \geq p_{jk}^{(m)} p_{kk}^{(r)} p_{kj}^{(n)}.$$

This is true because the left side of the inequality is the probability of the event $A = \{X_0 = j, X_{m+r+n} = j\}$ and the right side is probability of the event $B = \{X_0 = j, X_m = k, X_{r+n} = k, X_{m+r+n} = j\}$. Then it is clear that $B \subseteq A$ and thus we get the above inequality.

Summing over every $r \geq 0$, we get that:

$$\sum_{r=0}^{\infty} p_{jj}^{(m+r+n)} \geq \sum_{r=0}^{\infty} p_{jk}^{(m)} p_{kk}^{(r)} p_{kj}^{(n)}$$

$$\Rightarrow \sum_{r=0}^{\infty} p_{kk}^{(r)} \leq \frac{1}{p_{jk}^{(m)} p_{kj}^{(n)}} \sum_{r=0}^{\infty} p_{jj}^{(m+r+n)} \leq \frac{1}{p_{jk}^{(m)} p_{kj}^{(n)}} \sum_{l=0}^{\infty} p_{jj}^{(l)} < \infty.$$

The last inequality holds because we assume j to be **transient** and therefore the sum must be finite by theorem (1). Then, k must be transient as well by theorem (1).

It immediately follows that any state j is **recurrent** if and only if every other state in the communication class of j is **recurrent**. In particular, we notice that if a MC is irreducible, then the whole system must either be transient or recurrent.

<u>**Def**. (Regular chain)</u>

An irreducible or recurrent MC is called a <u>**regular**</u> chain if some power of the TPM $\mathbf{M}$ has only positive elements. The easiest way to check regularity is to keep track of whether the entries in the powers of $\mathbf{M}$ are positive. This can be done without computing numerical values by putting an x in the entry if it is positive and a 0 otherwise. To check regularity, let $SS=\{1,2,3,4\}$:

$$\mathbf{M}=\left(p_{ij}\right)_{i,j\in SS}=\begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}0 & x & 0 & 0\\ x & x & x & 0\\ 0 & x & x & x\\ 0 & 0 & x & 0\end{pmatrix},\ \mathbf{M}^2=\mathbf{M}\times\mathbf{M}=\begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}x & x & x & 0\\ x & x & x & x\\ x & x & x & x\\ 0 & x & x & x\end{pmatrix}, \text{ and}$$

$$\mathbf{M}^4=\mathbf{M}^2\times\mathbf{M}^2=\begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}x & x & x & x\\ x & x & x & x\\ x & x & x & x\\ x & x & x & x\end{pmatrix}.$$

Since all entries in $\mathbf{M}^4$ are **positive**, the chain is **regular**.

Note that the test for regularity is made faster by squaring the result each time.

Example: Consider a MC on state space $SS=\{1,2,3,4\}$ with TPM:

$$\mathbf{M}=\left(p_{ij}\right)_{i,j\in SS}=\begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}0 & 1 & 0 & 0\\ 1/3 & 0 & 2/3 & 0\\ 0 & 2/3 & 0 & 1/3\\ 0 & 0 & 1 & 0\end{pmatrix}.$$ To check **regularity**, let

$$\mathbf{M}=\left(p_{ij}\right)_{i,j\in SS}=\begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}0 & x & 0 & 0\\ x & 0 & x & 0\\ 0 & x & 0 & x\\ 0 & 0 & x & 0\end{pmatrix},\ \mathbf{M}^2=\mathbf{M}\times\mathbf{M}=\begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}x & 0 & x & 0\\ 0 & x & 0 & x\\ x & 0 & x & 0\\ 0 & x & 0 & x\end{pmatrix}, \text{ and}$$

$$\mathbf{M}^4 = \mathbf{M}^2 \times \mathbf{M}^2 = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} x & 0 & x & 0 \\ 0 & x & 0 & x \\ x & 0 & x & 0 \\ 0 & x & 0 & x \end{pmatrix}.$$

Observe that even powers of $\mathbf{M}$ will have $0's$ in the even numbered entries of row 1. Furthermore,

$$\mathbf{M}^3 = \mathbf{M}^2 \times \mathbf{M} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & x & 0 & x \\ x & 0 & x & 0 \\ 0 & x & 0 & x \\ x & 0 & x & 0 \end{pmatrix}, \text{ and } \mathbf{M}^5 = \mathbf{M}^2 \times \mathbf{M}^3 = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & x & 0 & x \\ x & 0 & x & 0 \\ 0 & x & 0 & x \\ x & 0 & x & 0 \end{pmatrix}.$$

Note that odd powers of $\mathbf{M}$ will have $0's$ in the odd numbered entries of row 0. This chain is **not regular** because no power of the transition matrix has only positive elements. This example has demonstrated that a **periodic chain cannot be regular**. Hence, a regular MC is irreducible and aperiodic.

Example (4).(Four-State Model of Weather)

Suppose that weather can be classified as either **raining**, **snowing**, **cloudy**, or **sunny**. Observations of the weather are made at the same time every day. The daily weather is assumed to have the Markov property, which means that the weather tomorrow depends only on the weather today. That is, the weather yesterday or on prior days will not affect the weather tomorrow. Since the daily weather is assumed to have the Markov property, the weather will be modeled as a MC with four states. The state X_n denotes the weather on day n for $n = 0,1,2,...$. The states are indexed below:

State, X_n	Description
1	Raining
2	Snowing
3	Cloudy
4	Sunny

The state space is $SS = \{1,2,3,4\}$.

Transition probabilities

Transition probabilities are based on the following observations. If it is raining today, the probabilities that tomorrow will bring rain, snow, clouds, or sun are 0.3, 0.1, 0.4, and 0.2, respectively. If it is snowing today, the probabilities of rain, snow, clouds, or sun tomorrow are 0.2, 0.5, 0.2, and 0.1, respectively. If today is cloudy, the probabilities that rain, snow, clouds, or sun will appear tomorrow are 0.3, 0.2, 0.1, and 0.4, respectively. Finally, if today is sunny, the probabilities that tomorrow it will be snowing, cloudy, or sunny are 0.6, 0.3, and 0.1, respectively. (A sunny day is never followed by a rainy day). Transition probabilities are obtained in the following manner. If day n designates today, then day $n+1$ designates tomorrow. If X_n designates the state today, then X_{n+1} designates the state tomorrow. If it is raining today, then $X_n = 1$. If it is cloudy tomorrow, then $X_{n+1} = 3$. Consider the four-state MC model of the weather on state space $SS = \{1,2,3,4\}$ with

TPM $\mathbf{M}$:

$$\mathbf{M} = \left(p_{ij}\right)_{i,j\in SS} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0.3 & 0.1 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0 & 0.6 & 0.3 & 0.1 \end{pmatrix} \end{matrix}. \qquad \text{(i)}$$

The two-step TPM is

$$\mathbf{M}^{(2)} = \mathbf{M}^2 = \mathbf{M} \times \mathbf{M} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0.23 & 0.28 & 0.24 & 0.25 \\ 0.22 & 0.37 & 0.23 & 0.18 \\ 0.16 & 0.39 & 0.29 & 0.16 \\ 0.21 & 0.42 & 0.18 & 0.19 \end{pmatrix}.$$

The four-step TPM is

$$\mathbf{M}^{(4)} = \mathbf{M}^4 = \mathbf{M}^2 \times \mathbf{M}^2 = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0.2054 & 0.3666 & 0.2342 & 0.1938 \\ 0.2066 & 0.3638 & 0.2370 & 0.1926 \\ 0.2026 & 0.3694 & 0.2410 & 0.1870 \\ 0.2094 & 0.3642 & 0.2334 & 0.1930 \end{pmatrix}.$$

The eight-step TPM is computed as follows:

$$\mathbf{M}^{(8)} = \mathbf{M}^8 = \mathbf{M}^4 \times \mathbf{M}^4 = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0.2060 & 0.3658 & 0.2367 & 0.1916 \\ 0.2059 & 0.3658 & 0.2367 & 0.1916 \\ 0.2059 & 0.3658 & 0.2467 & 0.1816 \\ 0.2060 & 0.3658 & 0.2367 & 0.1916 \end{pmatrix}.$$

Observe that as the exponent n increases from 1 to 2, from 2 to 4, and from 4 to 8, the entries of $\mathbf{M}^{(n)} = \left(p_{ij}^{(n)}\right)_{i,j\in SS}$ approach limiting values. When $n = 8$, all the rows of $\mathbf{M}^{(8)}$ are almost identical. One may infer that as n becomes very large, all the rows of $\mathbf{M}^{(n)}$ approach the same stationary probability vector, namely

$$\mathbf{P}^{(n)} = \begin{pmatrix} p_1^{(n)} & p_2^{(n)} & p_3^{(n)} & p_4^{(n)} \end{pmatrix} = \begin{pmatrix} 0.2059 & 0.3658 & 0.2367 & 0.1916 \end{pmatrix}.$$

That is, after n transitions, as n becomes very large, the n-step transition probability $p_{ij}^{(n)}$ approaches a limiting probability, $p_j^{(n)}$, irrespective of the starting state i.

If π_j denotes the **limiting probability for state** j in an L-state MC, then the limiting probability is defined by the formula

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}, \text{ for } j = 1, 2, ..., L$$

The limiting probability π_j is called a **steady-state probability**.

The vector of steady-state probabilities for an L-state MC is a $1\times L$ row vector denoted by $\boldsymbol{\pi} = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_L \end{pmatrix}$.

Since $\boldsymbol{\pi}$ is a probability vector, the entries of $\boldsymbol{\pi}$ must sum to one. Thus,

$$\pi_j > 0 \text{ for } j = 1, ... L \quad \text{and,} \quad \sum_{j=1}^{L} \pi_j = 1. \qquad \text{(a)}$$

This is called the **normalizing equation**.

The behavior of $\mathbf{M}^{(n)}$ for the **four-state regular** MC suggests that as $n\to\infty$, $\mathbf{M}^{(n)}$ will converge to a matrix $\mathbf{\Pi}$ with identical rows. Each row of $\mathbf{\Pi}$ is equal to the steady-state probability vector, $\boldsymbol{\pi}$:

$$\lim_{n\to\infty}\mathbf{M}^{(n)} = \lim_{n\to\infty}\mathbf{M}^{n} = \mathbf{\Pi} = \begin{matrix} 1 \\ 2 \\ \vdots \\ L \end{matrix}\begin{pmatrix} \boldsymbol{\pi} \\ \boldsymbol{\pi} \\ \vdots \\ \boldsymbol{\pi} \end{pmatrix} = \begin{matrix} 1 \\ 2 \\ \vdots \\ L \end{matrix}\begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_L \\ \pi_1 & \pi_2 & & \pi_L \\ \vdots & \vdots & \vdots & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_L \end{pmatrix}.$$

Thus, $\mathbf{\Pi}$ is a matrix with each row $\boldsymbol{\pi}$ is equal to **the steady-state probability vector**.

For the four-state MC model of the **weather**, the rows of $\mathbf{M}^{(8)}$

calculated above indicate that

$$\boldsymbol{\pi} \approx \begin{pmatrix} 0.2059 & 0.3658 & 0.2367 & 0.1916 \end{pmatrix}. \qquad \text{(ii)}$$

For large n, the state probability $p_j^{(n)}$ approaches the limiting probability π_j. That is,

$$\pi_j = \lim_{n\to\infty} p_j^{(n)} = \lim_{n\to\infty} p_{ij}^{(n)}, \text{ for } j = 1, 2, \ldots, L,$$

and does not depend on the **starting state**. Thus, the vector $\boldsymbol{\pi}$ of steady-state probabilities is equal to the limit, as the number of transitions approaches infinity, of the vector $\mathbf{P}^{(n)}$ of state probabilities. That is, $\boldsymbol{\pi} = \lim_{n\to\infty} \mathbf{P}^{(n)}$.

<u>Steady-State Probabilities for a Four-State Model of Weather</u>

Two approaches to solving the steady-state equations will be illustrated by applying them to the four-state regular MC model of the weather for which the TPM is given as in formula (i).

<u>The first approach</u>.

The matrix form of the steady-state equations is

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} 0.3 & 0.1 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0 & 0.6 & 0.3 & 0.1 \end{pmatrix}, \qquad \text{(iii)}$$

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1.$$

In algebraic form the system of steady-state equations (iii) of five equations in four unknowns is produced:

$$\begin{aligned}
\pi_1 &= (0.3)\pi_1 + (0.2)\pi_2 + (0.3)\pi_3 + (0)\pi_4 \\
\pi_2 &= (0.1)\pi_1 + (0.5)\pi_2 + (0.2)\pi_3 + (0.6)\pi_4 \\
\pi_3 &= (0.4)\pi_1 + (0.2)\pi_2 + (0.1)\pi_3 + (0.3)\pi_4 . \qquad \text{(iv)} \\
\pi_4 &= (0.2)\pi_1 + (0.1)\pi_2 + (0.4)\pi_3 + (0.1)\pi_4 \\
\pi_1 &+ \pi_2 + \pi_3 + \pi_4 = 1
\end{aligned}$$

In this approach the fourth equation is arbitrarily deleted, and the resulting system of four equations in four unknowns is shown below:

$$\begin{aligned}
\pi_1 &= (0.3)\pi_1 + (0.2)\pi_2 + (0.3)\pi_3 + (0)\pi_4 \\
\pi_2 &= (0.1)\pi_1 + (0.5)\pi_2 + (0.2)\pi_3 + (0.6)\pi_4 \\
\pi_3 &= (0.4)\pi_1 + (0.2)\pi_2 + (0.1)\pi_3 + (0.3)\pi_4 \\
\pi_1 &+ \pi_2 + \pi_3 + \pi_4 = 1
\end{aligned}$$

The solution of this system is $\boldsymbol{\pi} = (\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4)$

$$= (1407/6832 \quad 2499/6832 \quad 1617/6832 \quad 1309/6832)$$

$$= (0.2059 \quad 0.3658 \quad 0.2367 \quad 0.1916). \qquad \text{(b)}$$

This solution almost matches the approximate one obtained in formula (ii) by calculating $\mathbf{M}^{(8)}$.

The second approach.

The normalizing equation (a) is initially **ignored**. The first three equations contained in the system

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{M}$$

are solved to express π_1, π_2, and π_3 as the following constants times π_4: $\pi_1 = (1407/1309)\pi_4$, $\pi_2 = (2499/1309)\pi_4$, and $\pi_3 = (1617/1309)\pi_4$.

These values for π_1, π_2, and π_3 expressed in terms of π_4 are substituted into the normalizing equation to solve for π_4.

$$1 = \pi_1 + \pi_2 + \pi_3 + \pi_4,$$
$$1 = (1407/1309)\pi_4 + (2499/1309)\pi_4 + (1617/1309)\pi_4 + \pi_4.$$

The result is $\pi_4 = (1309/6832)$.

Substituting the result for π_4 to solve for the other steady-state probabilities gives the values obtained by following the **first approach**:

$$\pi_1 = (1407/6832),\ \pi_2 = (2499/6832), \text{ and } \pi_3 = (1617/6832).$$

The steady-state probability π_i represents the long run proportion of time that the weather will be represented by state i. For example, the long run proportion of cloudy days is equal to

$$\pi_3 = (1617/6832) = 0.2367.$$

Probabilities of n-step first passage

To determine the probability that the first visit to state $j = 1$ will occur at time n, when the initial state is $i \in SS = \{1,2,3,4\}$, in the four-state regular MC **model of the weather**. The TPM is given as in formula (i). If the state $j = 1$ (rain) is **the target state**, then

$$\mathbf{f}_1^{(1)} = \left(f_{i1}^{(n)}\right)_{i \in SS} = \left(P_i(T_1 = n)\right)_{i \in SS} = \begin{pmatrix} f_{11}^{(1)} \\ f_{21}^{(1)} \\ f_{31}^{(1)} \\ f_{41}^{(1)} \end{pmatrix} = \begin{pmatrix} p_{11} \\ p_{21} \\ p_{41} \\ p_{41} \end{pmatrix} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}\begin{pmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.0 \end{pmatrix}, \text{ and}$$

$$\mathbb{Z} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 0.1 & 0.4 & 0.2 \\ 0 & 0.5 & 0.2 & 0.1 \\ 0 & 0.2 & 0.1 & 0.4 \\ 0 & 0.6 & 0.3 & 0.1 \end{pmatrix}.$$

The vectors of n-step first passage probabilities, for $n = 2, 3$, and 4, are calculated below, along with $\mathbb{Z}^{n-1}$:

$$\mathbf{f}_1^{(2)} = \mathbb{Z}\mathbf{f}_1^{(1)} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 0.1 & 0.4 & 0.2 \\ 0 & 0.5 & 0.2 & 0.1 \\ 0 & 0.2 & 0.1 & 0.4 \\ 0 & 0.6 & 0.3 & 0.1 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.0 \end{pmatrix} = \begin{pmatrix} 0.14 \\ 0.16 \\ 0.07 \\ 0.21 \end{pmatrix} = \begin{pmatrix} f_{11}^{(2)} \\ f_{21}^{(2)} \\ f_{31}^{(2)} \\ f_{41}^{(2)} \end{pmatrix},$$

$$\mathbf{f}_1^{(3)} = \mathbb{Z}\mathbf{f}_1^{(2)} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 0.1 & 0.4 & 0.2 \\ 0 & 0.5 & 0.2 & 0.1 \\ 0 & 0.2 & 0.1 & 0.4 \\ 0 & 0.6 & 0.3 & 0.1 \end{pmatrix} \begin{pmatrix} 0.14 \\ 0.16 \\ 0.07 \\ 0.21 \end{pmatrix} = \begin{pmatrix} 0.086 \\ 0.115 \\ 0.123 \\ 0.138 \end{pmatrix} = \begin{pmatrix} f_{11}^{(3)} \\ f_{21}^{(3)} \\ f_{31}^{(3)} \\ f_{41}^{(3)} \end{pmatrix}.$$

Alternatively,

$$\mathbf{f}_1^{(3)} = \mathbb{Z}^2 \mathbf{f}_1^{(1)} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 0.25 & 0.12 & 0.19 \\ 0 & 0.35 & 0.15 & 0.14 \\ 0 & 0.36 & 0.17 & 0.10 \\ 0 & 0.42 & 0.18 & 0.19 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.0 \end{pmatrix} = \begin{pmatrix} 0.086 \\ 0.115 \\ 0.123 \\ 0.138 \end{pmatrix} = \begin{pmatrix} f_{11}^{(3)} \\ f_{21}^{(3)} \\ f_{31}^{(3)} \\ f_{41}^{(3)} \end{pmatrix}.$$

$$\mathbf{f}_1^{(4)} = \mathbb{Z}\mathbf{f}_1^{(3)} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 0.1 & 0.4 & 0.2 \\ 0 & 0.5 & 0.2 & 0.1 \\ 0 & 0.2 & 0.1 & 0.4 \\ 0 & 0.6 & 0.3 & 0.1 \end{pmatrix} \begin{pmatrix} 0.086 \\ 0.115 \\ 0.123 \\ 0.138 \end{pmatrix} = \begin{pmatrix} 0.0883 \\ 0.0959 \\ 0.0905 \\ 0.1197 \end{pmatrix} = \begin{pmatrix} f_{11}^{(4)} \\ f_{21}^{(4)} \\ f_{31}^{(4)} \\ f_{41}^{(4)} \end{pmatrix},$$

Alternatively,

$$\mathbf{f}_1^{(4)} = \mathbb{Z}^3 \mathbf{f}_1^{(1)} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 0.263 & 0.119 & 0.092 \\ 0 & 0.289 & 0.127 & 0.109 \\ 0 & 0.274 & 0.119 & 0.114 \\ 0 & 0.360 & 0.159 & 0.133 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.0 \end{pmatrix} = \begin{pmatrix} 0.0883 \\ 0.0959 \\ 0.0905 \\ 0.1197 \end{pmatrix} = \begin{pmatrix} f_{11}^{(4)} \\ f_{21}^{(4)} \\ f_{31}^{(4)} \\ f_{41}^{(4)} \end{pmatrix}.$$

The probability that the chain moves from state 4 to target state 1 for the first time in **4-steps** is given by $f_{41}^{(4)} = 0.1197$. Therefore, the probability that the next rainy day (state 1) will appear for the first time 4 days after a **sunny day** (state 4) **is** 0.1197.

Mean recurrence time for weather MC model

The first method .

The following additional equation can be solved to calculate h_{11}, the mean recurrence time for state $j = 1$:

$$h_{11} = 1 + \begin{pmatrix} p_{12} & p_{13} & p_{14} \end{pmatrix} \begin{pmatrix} h_{21} \\ h_{31} \\ h_{41} \end{pmatrix}.$$

Then $h_{11} = 1 + p_{12}h_{21} + p_{13}h_{31} + p_{14}h_{41}$

$$= 1 + (0.1)h_{21} + (0.4)h_{31} + (0.2)h_{41}$$

$$= 1 + (0.1)(5.3234) + (0.4)(5.1244) + (0.2)(6.3682) = 4.85574 .$$

The second method.

If the steady probabilities are known, the mean recurrence time h_{11} for a target state $j = 1$ is simply the **reciprocal** of the steady-state probability π_1 for the target state $j = 1$.

The **steady-state probability vector for the four-state regular**

MC for which the TPM is given as in formula (i), is obtained as in formula (b):

$$\boldsymbol{\pi} = (\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4) = (0.2059 \quad 0.3658 \quad 0.2367 \quad 0.1916). \quad \text{(b)}$$

Observe that $\pi_1 = 0.2059$.

Hence, the **mean recurrence time for state** $j = 1$ is,

$$[h_{11} = 1/\pi_1 = 1/0.2059 = 4.85574],$$

which is **close to** the result obtained by the **first method**. Discrepancies are due to roundoff error.