المحاضرة الثامنة

الفرقة: الثالثة

الشعبة: الرياضيات

المادة: نظرية المعادلات التفاضلية

In Exercises 7–22, determine the eigenvalues and eigenvectors if the eigenvalues are real. Also classify the system (state whether stable or unstable node, stable or unstable spiral, center, saddle point) and in all cases sketch the phase plane of the linear system. (As a hint, problems with * have complex eigenvalues.)

7. $\frac{dx}{dt} = x, \frac{dy}{dt} = x + 2y.$

8. $\frac{dx}{dt} = 2x - y, \frac{dy}{dt} = 3x - 2y.$

9*. $\frac{dx}{dt} = -x - 5y, \frac{dy}{dt} = x + y.$

10. $\frac{dx}{dt} = 2x - y, \frac{dy}{dt} = 2x + 5y.$

11*. $\frac{dx}{dt} = x - y, \frac{dy}{dt} = x + y.$

12*. $\frac{dx}{dt} = -2x + 2y, \frac{dy}{dt} = -x.$

13. $\frac{dx}{dt} = -5x - 4y, \frac{dy}{dt} = 2x + y.$

14*. $\frac{dx}{dt} = x + 5y, \frac{dy}{dt} = -2x - y.$

15. $\frac{dx}{dt} = y, \frac{dy}{dt} = 2x + y.$

16*. $\frac{dx}{dt} = -x - 2y$, $\frac{dy}{dt} = 2x - y$.

17. $\frac{dx}{dt} = -5x - y$, $\frac{dy}{dt} = 3x - y$.

18*. $\frac{dx}{dt} = x + 2y$, $\frac{dy}{dt} = -4x - 3y$.

19*. $\frac{dx}{dt} = -x + 4y$, $\frac{dy}{dt} = -4x - y$.

20*. $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} = -2x + 3y$.

21. $\frac{dx}{dt} = 4x + 3y$, $\frac{dy}{dt} = 3x + 4y$.

22. $\frac{dx}{dt} = 2x + 3y$, $\frac{dy}{dt} = 3x + 2y$.

7. $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$

$\det \begin{bmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0$

$\Rightarrow \lambda = 1,\ 2$, both positive $\Rightarrow$ Unstable node.

$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}$ so that eigenvector, $\begin{bmatrix} u \\ v \end{bmatrix}$, satisfies

$\begin{bmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Eigenvector for $\lambda = 1$: $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u + v = 0 \Rightarrow$

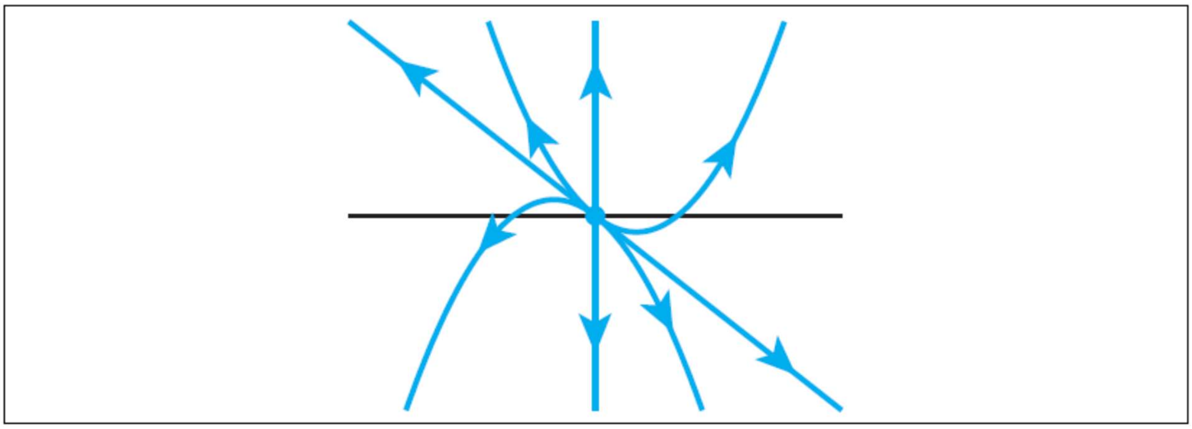
$v = 1$, $u = -1$, and eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Solution of the differential equation is $\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t}$.

Eigenvector for $\lambda = 2$: $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -u = 0 \Rightarrow$

$u = 0$, v is free to choose, say, $v = 1$ and eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution of the differential equation is $\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$.

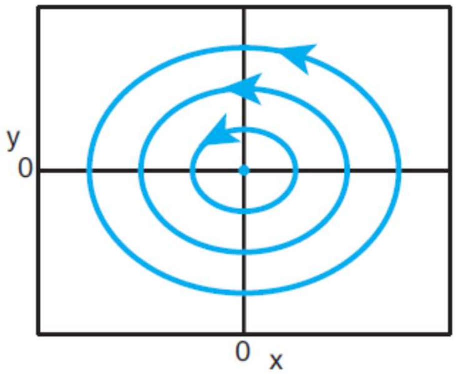
Phase plane:



9. $A = \begin{bmatrix} -1 & -5 \\ 1 & 1 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$

$\det \begin{bmatrix} -1-\lambda & -5 \\ 1 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (-1-\lambda)(1-\lambda) + 5 = 0$

$\Rightarrow \lambda^2 + 4 \Rightarrow \lambda = \pm 2i$, complex with 0 real part $\Rightarrow$Stable center.



11. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$

$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 + 1 = 0$

$\Rightarrow (1-\lambda)^2 = -1 \Rightarrow 1 - \lambda = \pm i \Rightarrow \lambda = 1 \pm i$, complex with positive real part $\Rightarrow$Unstable spiral.

13. $A = \begin{bmatrix} -5 & -4 \\ 2 & 1 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$

$\det \begin{bmatrix} -5-\lambda & -4 \\ 2 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (-5-\lambda)(1-\lambda) + 8 = 0$

$\Rightarrow \lambda^2 + 4\lambda + 3 = 0 \Rightarrow (\lambda + 1)(\lambda + 3) \Rightarrow \lambda = -1, -3$, both negative $\Rightarrow$ Stable node. $\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}$ so that eigenvector,

$\begin{bmatrix} u \\ v \end{bmatrix}$, satisfies $\begin{bmatrix} -5-\lambda & -4 \\ 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Eigenvector for $\lambda = -1$: $\begin{bmatrix} -4 & -4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

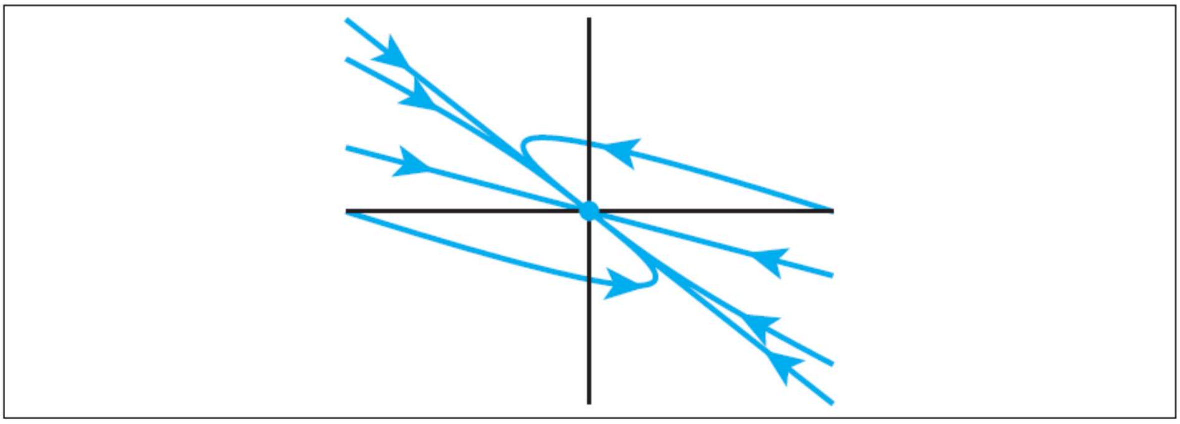
$\Rightarrow -4u - 4v = 0 \Rightarrow v = 1,\ u = -1$ and eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solution of the differential equation is $\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$.

Eigenvector for $\lambda = -3$: $\begin{bmatrix} -2 & -4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2u - 4v = 0 \Rightarrow v = 1,\ u = -2$, and eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Solution of the differential equation is $\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t}$.



15. $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$

$\det \begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} = 0 \Rightarrow -\lambda(1-\lambda) - 2 = 0$

$\Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda+1)(\lambda-2) \Rightarrow \lambda = -1,\ 2$, one positive and one negative $\Rightarrow$Unstable saddle point. $\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}$ so that eigenvector, $\begin{bmatrix} u \\ v \end{bmatrix}$, satisfies $\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

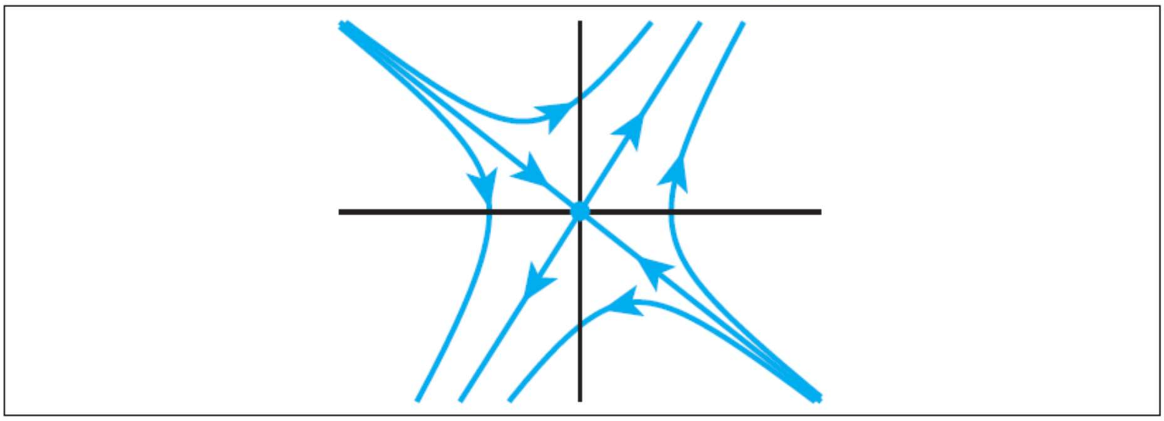
Eigenvector for $\lambda = -1$: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u + v = 0 \Rightarrow$

$u = 1,\ v = -1$ and eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Solution of the differential equation is $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$.

Eigenvector for $\lambda = 2$: $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2u + v = 0 \Rightarrow u = 1,\ v = 2$, and eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution of the differential equation is $\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$.



19. $\left.\begin{array}{l}\frac{dx}{dt} = -x + 4y \\ \frac{dy}{dt} = -4x - y\end{array}\right\} \Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t}\begin{bmatrix} u \\ v \end{bmatrix}$ where λ, $\begin{bmatrix} u \\ v \end{bmatrix}$ are eigenvalue and corresponding eigenvector, respectively, of $A = \begin{bmatrix} -1 & 4 \\ -4 & -1 \end{bmatrix}$.

The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$

$\det \begin{bmatrix} -1-\lambda & 4 \\ -4 & -1-\lambda \end{bmatrix} = 0 \Rightarrow (-1-\lambda)^2 + 16 = 0$

$\Rightarrow (1+\lambda)^2 = -16 \Rightarrow 1 + \lambda = \pm 4i \Rightarrow \lambda = -1 \pm 4i$,

complex with negative real part $\Rightarrow$Stable spiral.

21. $\left.\begin{array}{l}\frac{dx}{dt} = 4x + 3y \\ \frac{dy}{dt} = 3x + 4y\end{array}\right\} \Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t}\begin{bmatrix} u \\ v \end{bmatrix}$ where λ, $\begin{bmatrix} u \\ v \end{bmatrix}$ are eigenvalue and corresponding eigenvector, respectively, of $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$

$\det \begin{bmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{bmatrix} = 0 \Rightarrow (4-\lambda)^2 - 9 = 0$

$\Rightarrow \lambda^2 - 8\lambda + 7 = 0 \Rightarrow (\lambda - 1)(\lambda - 7) \Rightarrow \lambda = 1, 7$, both positive $\Rightarrow$Unstable node.

Eigenvector, $\begin{bmatrix} u \\ v \end{bmatrix}$, satisfies $\begin{bmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

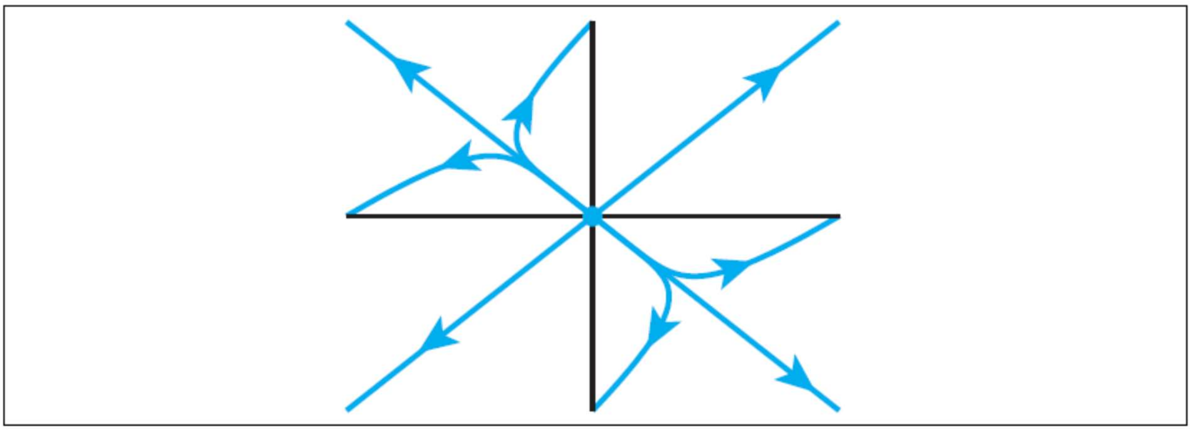
Eigenvector for $\lambda = 1$: $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3u + 3v = 0 \Rightarrow$

$v = 1$, $u = -1$ and eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Solution of the differential equation is $\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t$.

Eigenvector for $\lambda = 7$: $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3u + 3v = 0 \Rightarrow u = 1$, $v = 1$, and eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution of the differential equation is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}$.



In Exercises 23–35, determine the eigenvalues and eigenvectors if the eigenvalues are real, classify the system (state whether stable or unstable node, stable or unstable spiral, center, saddle point) and in all cases sketch the phase plane of the linear system. (As a hint, problems with * have complex eigenvalues.)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x},$$

Where

23*. $a=0, b=1, c=-4, d=0.$

24. $a=1, b=3, c=1, d=-1.$

25. $a=2, b=1, c=1, d=2.$

26*. $a=-3, b=-2, c=1, d=-5.$

27*. $a=3, b=-1, c=1, d=2.$

28. $a=-2, b=0, c=0, d=-3.$

29. $a=1, b=0, c=1, d=-3.$

30. $a=-1, b=3, c=1, d=1.$

31. $a=4, b=-3, c=1, d=0.$

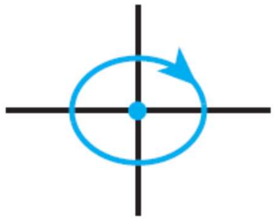
32*. $a=-1, b=2, c=-2, d=-1.$

33. $a=2, b=-1, c=1, d=0.$

34. $a=3, b=2, c=0, d=4.$

35. $a=1, b=0, c=0, d=-3.$

23. $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$ $\det \begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$, complex with 0 real part $\Rightarrow$Stable center.



25. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0$

$\Rightarrow \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 = 0$

$\Rightarrow 2 - \lambda = \pm 1 \Rightarrow \lambda = 3,\ 1$, both positive $\Rightarrow$ Unstable node.

$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}$ so that eigenvector, $\begin{bmatrix} u \\ v \end{bmatrix}$, satisfies

$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

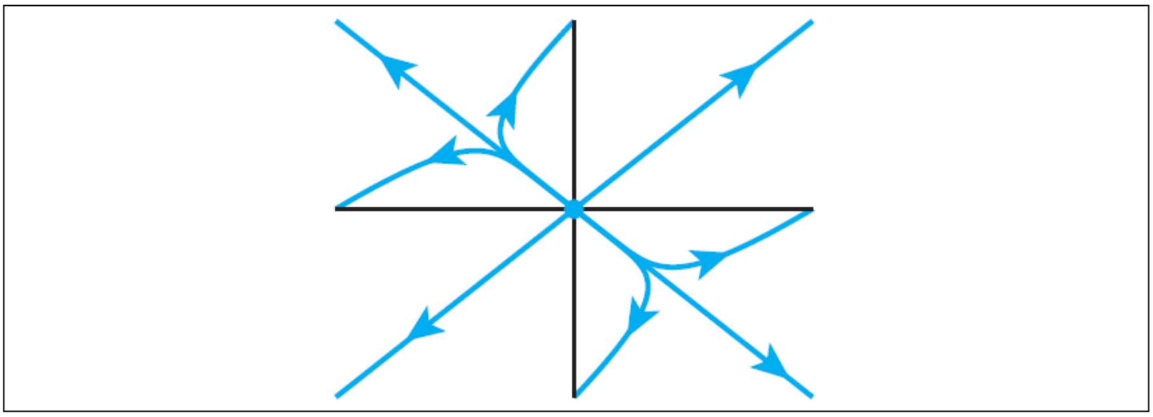
Eigenvector for $\lambda = 3$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -u + v = 0 \Rightarrow u = 1,\ v = 1$ and eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

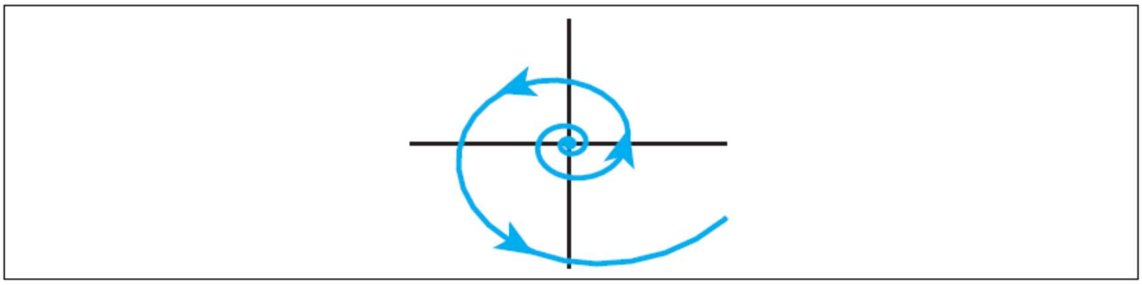
Solution of the differential equation is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$.

Eigenvector for $\lambda = 1$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u + v = 0 \Rightarrow$

$u = 1,\ v = -1$ and eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Solution of the differential equation is $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$.



27. $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0 \Rightarrow$
$\det \begin{bmatrix} 3-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda) + 1 = 0$
$\Rightarrow \lambda^2 - 5\lambda + 7 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{25-28}}{2} = \frac{5}{2} \pm \frac{\sqrt{3}}{2} i$, complex with positive real part $\Rightarrow$ Unstable spiral.



29. $A = \begin{bmatrix} 1 & 0 \\ 1 & -3 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0$

$\Rightarrow \det \begin{bmatrix} 1-\lambda & 0 \\ 1 & -3-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)(-3-\lambda) = 0$

$\Rightarrow \lambda = 1, -3$, one positive and one negative

$\Rightarrow$ Unstable saddle point. $\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}$ so that eigenvector,

$\begin{bmatrix} u \\ v \end{bmatrix}$, satisfies $\begin{bmatrix} 1-\lambda & 0 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

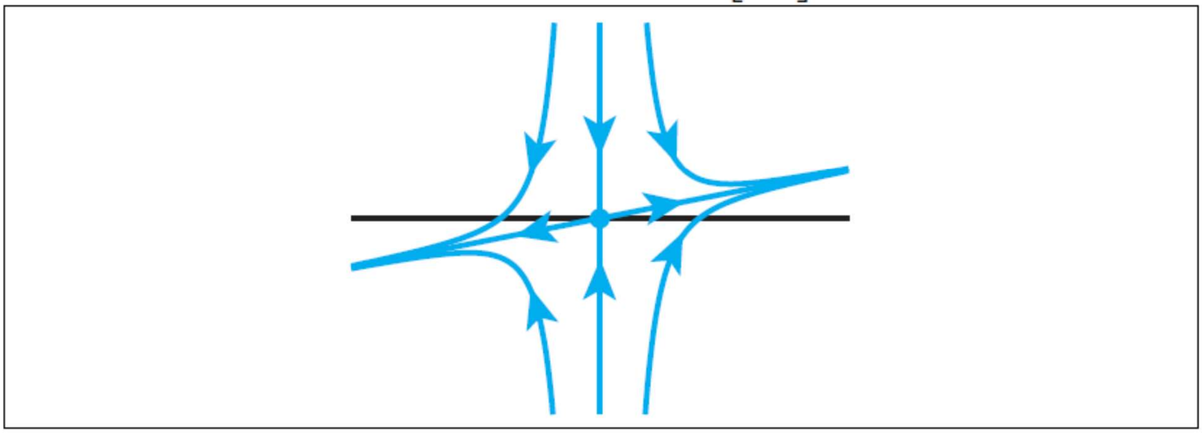
Eigenvector for $\lambda = 1$: $\begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u - 4v = 0 \Rightarrow$

$v = 1, u = 4$ and eigenvector is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Solution of the differential equation is $\begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t$.

Eigenvector for $\lambda = -3$: $\begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4u = 0$

$u = 0$, v is free to choose, say, $v = 1$ and eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution of the differential equation is $\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}$.



31. $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$. The eigenvalue, λ, satisfies $\det(A - \lambda I) = 0$

$\Rightarrow \det \begin{bmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{bmatrix} = 0 \Rightarrow -\lambda(4-\lambda) + 3 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda - 1)(\lambda - 3) = 0 \Rightarrow \lambda = 1,\ 3$, both positive

$\Rightarrow$Unstable node. $\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u \\ v \end{bmatrix}$ so that eigenvector,

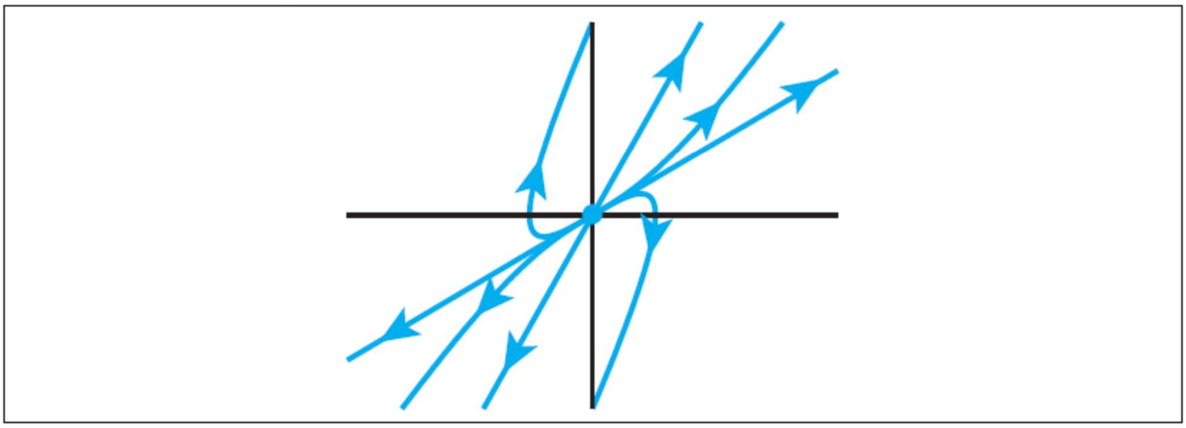
$\begin{bmatrix} u \\ v \end{bmatrix}$, satisfies $\begin{bmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Eigenvector for $\lambda = 1$: $\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3u - 3v = 0 \Rightarrow$

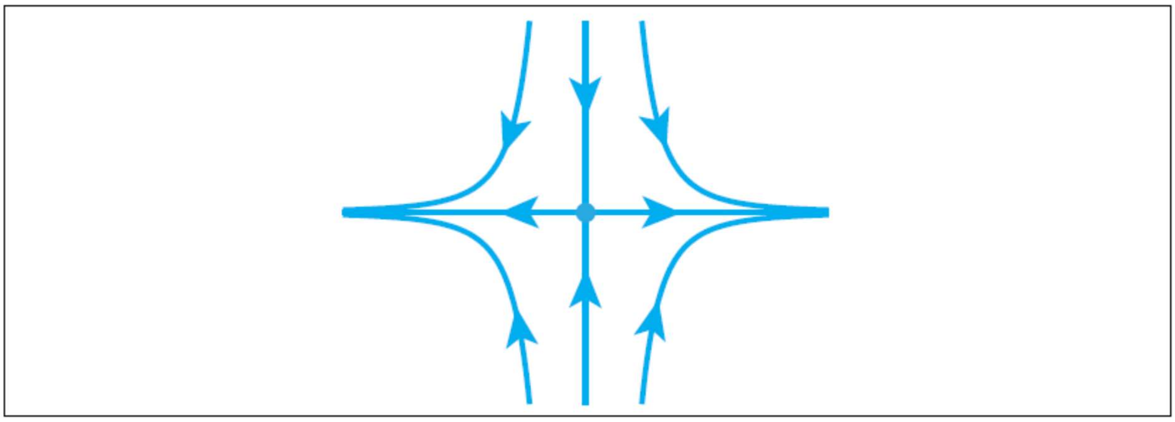
$u = 1, v = 1$ and eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Solution of the differential equation is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$.

Eigenvector for $\lambda = 3$: $\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u - 3v = 0$

$u = 1,\ v = 3$ and eigenvector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Solution of the differential equation is $\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$.



35)



For Exercises 29–35 without finding the eigenvalues, classify the system (stable or unstable node, stable or unstable spiral, center, saddle point), determine using the trace and determinant condition.

(29) $A = \begin{bmatrix} 1 & 0 \\ 1 & -3 \end{bmatrix}$. $\det \mathbf{A} = -3$, $tr\mathbf{A} = -2$.

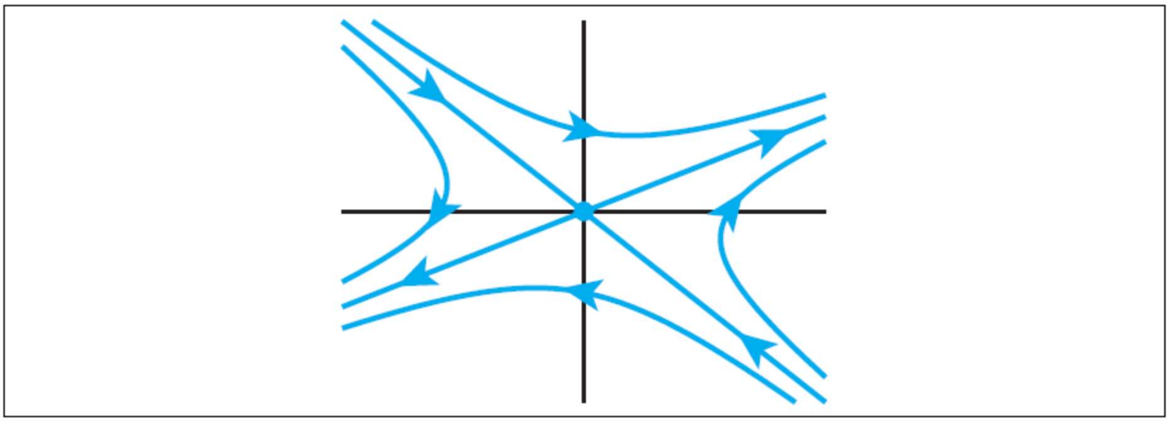
Now $4 \det \mathbf{A} = -12 < 4 = (tr\mathbf{A})^2 \Rightarrow$ eigenvalues are real.
Also $\det \mathbf{A} < 0$ implies unstable saddle point.

(35) $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$. $\det \mathbf{A} = -3$, $tr\mathbf{A} = -2$.

Now $4 \det \mathbf{A} = -12 < 4 = (tr\mathbf{A})^2 \Rightarrow$ eigenvalues are real.
Also $\det \mathbf{A} < 0$ implies unstable saddle point.

In Exercises 38–41, graph the phase portrait given the eigenvalues and the eigenvectors:

39.



41.