المحاضرة التاسعة

الفرقة: الثالثة

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المادة: نظرية المعادلات التفاضلية

Nonlinear Systems of Differential Equations in the Plane

6.1 Introduction

In Chapter 5, we introduced the analysis of a single first-order differential equation $\frac{dx}{dt} = f(x)$. We discussed equilibrium and the stability of the equilibrium in that chapter. The current chapter will discuss similar questions for systems of **autonomous differential equations** of the form

$$\frac{dx}{dt} = f(x, y), \tag{1a}$$

$$\frac{dy}{dt} = g(x, y). \tag{1b}$$

This chapter is self-contained in the sense that the material from Chapter 5 is not necessary. However, Chapter 4 is essential for our presentation, in which we discussed the linear systems

$$\frac{dx}{dt} = ax + by \tag{2a}$$

$$\frac{dy}{dt} = cx + dy \tag{2b}$$

and the phase plane for linear systems. In Section 6.2, we will show that the phase plane near an equilibrium for the nonlinear system (1a)–(1b) usually looks like the phase plane for the corresponding linear system.

6.2 Equilibria of Nonlinear Systems, Linear Stability Analysis of Equilibrium, and the Phase Plane

In this section, we study the nonlinear autonomous system of differential equations

$$\frac{dx}{dt} = f(x, y), \tag{1a}$$

$$\frac{dy}{dt} = g(x, y). \tag{1b}$$

Examples have been given in the introduction to this chapter.

Equilibrium Solutions

If $x(t)$, $y(t)$ is an **equilibrium** or constant solution of (1a)–(1b), then $x(t) = r$, $y(t) = s$ for constants r, s. Substituting into (1a)–(1b), we find that the equilibrium solutions r, s must satisfy

$$0 = f(r, s), \tag{2a}$$
$$0 = g(r, s). \tag{2b}$$

These nonlinear algebraic equations may be used to determine the equilibrium for a given system of differential equations. For linear systems $x = 0$, $y = 0$ is an equilibrium, so that is not much of an issue.

Example 6.2.1 *Equilibrium*

Find all equilibria of

$$\frac{dx}{dt} = -x + xy, \tag{3a}$$
$$\frac{dy}{dt} = -4y + 8xy. \tag{3b}$$

• SOLUTION. Let $x(t) = r$, $y(t) = s$, where r, s are constants. Thus (3a)–(3b) becomes

$$0 = -r + rs = r(-1 + s), \tag{4a}$$
$$0 = -4s + 8rs = 4s(-1 + 2r). \tag{4b}$$

Equation (4a) implies $r = 0$ or $s = 1$. Then

$$\text{If } r = 0, \text{ (4b) implies that } s = 0.$$
$$\text{If } s = 1, \text{ (4b) implies } -1 + 2r = 0, \text{ so that } r = \tfrac{1}{2}$$

Thus there are two equilibria for (3a)–(3b):

$$x = 0, y = 0 \quad \text{and} \quad x = \tfrac{1}{2}, y = 1. \tag{5}$$

♦

6.2.1 Linear Stability Analysis and the Phase Plane

Once an equilibrium is found, we will develop here ideas that determine whether the equilibrium is stable or unstable. We now wish to determine the behavior of the solutions of (1a) and (1b) near an equilibrium. Suppose that $x(t) = r$, $y(t) = s$ is an equilibrium. In Chapter 5 we analyzed first-order nonlinear autonomous differential equations near an equilibrium and used a Taylor series of one variable to determine the stability of an equilibrium. For our system (1a)–(1b), we use the two-dimensional version of Taylor's approximations learned in calculus, and we get

$$f(x, y) \approx f(r, s) + f_x(r, s)(x - r) + f_y(r, s)(y - s), \tag{6a}$$
$$g(x, y) \approx g(r, s) + g_x(r, s)(x - r) + g_y(r, s)(y - s), \tag{6b}$$

which is **only valid as an approximation near the equilibrium** $x = r$, $y = s$. This provides the linearization of a function of two variables. It is very important that the partial derivatives are evaluated at the equilibrium, and that they are just constants. The subscript notation for partial derivatives has again been used. For example, $f_x = \frac{\partial f(x,y)}{\partial x}$. The approximation (6) uses only the linear or first-order terms. A more accurate approximation can be gotten by using terms with higher powers of $x - r$ and $y - s$. We neglect these higher-order nonlinear terms for now and discuss later the importance of the neglected nonlinear terms. Since (r, s) is an equilibrium, it satisfies (2a) and (2b), $f(r, s) = 0$, $g(r, s) = 0$. This suggests that, near the equilibrium, the solutions of (1a) and (1b) can be approximated and resemble those of the following **linearized system of differential equations**:

$$\frac{dx}{dt} = f_x(r, s)(x - r) + f_y(r, s)(y - s), \tag{7a}$$

$$\frac{dy}{dt} = g_x(r, s)(x - r) + g_y(r, s)(y - s). \tag{7b}$$

We introduce the displacement from the equilibrium (as we did for first-order equations):

$$z = x - r, \tag{8a}$$

$$w = y - s \tag{8b}$$

(which translates the equilibrium point r, s to the origin $z = 0$, $w = 0$). In this way, since r and s are constants, (7a) and (7b) become

$$\frac{dz}{dt} = az + bw, \tag{9a}$$

$$\frac{dw}{dt} = cz + dw, \tag{9b}$$

where a, b, c, d are constants given by

$$a = f_x(r, s), b = f_y(r, s), c = g_x(r, s), d = g_y(r, s). \tag{10}$$

But (9a)–(9b) is a linear homogeneous system like that discussed in Chapter 4.

JACOBIAN MATRIX. Using matrix multiplication, the system (9a)–(9b) can be written as

$$\frac{d}{dt}\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} f_x(r, s) & f_y(r, s) \\ g_x(r, s) & g_y(r, s) \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}. \tag{11}$$

Matrix notation is particularly effective here, and we introduce the matrix of first derivatives called the **Jacobian matrix**:

$$\text{Jacobian matrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}. \tag{12}$$

The matrix must be evaluated at the equilibrium $x = r$, $y = s$. Introducing the vector displacement from equilibrium $\mathbf{z} = (z, w) = (x - r, y - s)$ whose components are z and w, equations (11) are more conveniently written using matrix notation:

$$\frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z}, \tag{13}$$

where $\mathbf{A}$ is a matrix of constants obtained by evaluating the Jacobian matrix at the equilibrium:

$$\mathbf{A} = \begin{bmatrix} f_x(r, s) & f_y(r, s) \\ g_x(r, s) & g_y(r, s) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{14}$$

The Phase Plane

The phase plane that we discussed with many examples in Chapter 4 for linear systems can also be introduced for nonlinear systems:

$$\frac{dx}{dt} = f(x, y), \tag{15a}$$

$$\frac{dy}{dt} = g(x, y). \tag{15b}$$

Orbits or trajectories can be introduced and solutions graphed in the x, y-plane. Some procedures for the phase plane of nonlinear systems will be discussed in the next subsection. Near an equilibrium, the nonlinear differential equation can be approximated by its associated linearized system. Thus, we expect that the phase plane for the nonlinear system near an equilibrium can be approximated in some sense by the phase plane of the corresponding linear system near the equilibrium. We will try to be fairly precise and state a theorem below. The claim we wish to make is that in most cases the phase plane for the nonlinear system in the neighborhood of an equilibrium resembles the phase plane for the corresponding linear system (the linearized system of differential equations defined above).

STABILITY OF AN EQUILIBRIUM AND PHASE PLANE NEAR AN EQUILIBRIUM. We wish to describe the relationship between the nonlinear system (15a)–(15b) near an equilibrium and its linearized system of differential equations (9a)–(9b). We are interested in the stability of an equilibrium, and we are interested in the phase plane of the nonlinear system in the neighborhood of an equilibrium. The following theorem is motivated but not proved from the linearized system.

THEOREM 6.2.1 *Suppose that (r, s) is an equilibrium of (15a) and (15b). Define a, b, c, d by (10) or (14). Let λ_1, λ_2 be the eigenvalues of the matrix A (14) which satisfy the characteristic equation derived from the determinant condition:*

$$\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - (a + d)\lambda + ad - bc = 0. \tag{16}$$

Recall that the eigenvalues λ of the matrix correspond to the time dependency $e^{\lambda t}$. To be more precise, we recall that the solutions of the linear system are linear *combinations of solutions $z = e^{\lambda t}v$, where v is an eigenvector of the matrix. Not all solutions are of this form. The stability of the equilibrium and the phase plane in the neighborhood of the equilibrium are usually determined by the linearization:*

1. *If $\lambda_1 > 0, \lambda_2 > 0$, the equilibrium of the nonlinear system is unstable, and the local phase plane will be an unstable node resembling figure 4.3.9 of Chapter 4.*
2. *If $\lambda_1 < 0, \lambda_2 < 0$, the equilibrium of the nonlinear system is asymptotically stable, and the local phase plane will be a stable node resembling figure 4.3.11 of Chapter 4.*
3. *If λ_1, λ_2 are nonzero and of opposite signs, the equilibrium is unstable, and the local phase plane will be a saddle point resembling figure 4.3.13 of Chapter 4.*
4. *If $\lambda_1 = \lambda_2 > 0$ ($\lambda_1 = \lambda_2 < 0$), then the equilibrium is unstable (stable), but we do not discuss the local phase plane.*
5. *If $\lambda_1 = 0$ and $\lambda_2 > 0$, then the equilibrium is unstable. If $\lambda_1 = 0$ and $\lambda_2 < 0$, then the equilibrium for the nonlinear system may be stable or unstable depending on the neglected nonlinear terms. We do not discuss the local phase plane in either case.*
6. *If $\lambda_1 = \alpha + i\delta, \lambda_2 = \alpha - i\delta$, with $\delta \neq 0$ and $\alpha > 0$, the equilibrium is unstable, and the local phase plane with be an unstable spiral (see a and b of figure 4.3.16 of Chapter 4.)*
7. *If $\lambda_1 = \alpha + i\delta, \lambda_2 = \alpha - i\delta$, with $\delta \neq 0$ and $\alpha < 0$, the equilibrium is asymptotically stable, and the local phase plane with be a stable spiral (see c and d of figure 4.3.16 of Chapter 4.)*
8. *If $\lambda_1 = i\delta, \lambda_2 = -i\delta$, then the linearization has oscillatory solutions. The phase plane of the linearization is a center (see figure 4.3.17 of Chapter 4). The phase plane of the nonlinear system often looks like the same center, but* **it is not guaranteed to look like a center.** *The equilibrium for the nonlinear system will be stable or unstable depending on the nonlinear terms neglected in the linearization. These will be discussed later. Only cases 5 and 8 have this potential difficulty.*

Local means that the linearized phase plane is valid only near the equilibrium of the nonlinear system.

Resembles means that the axis may be bent and the phase plane somewhat distorted in the neighborhood of the equilibrium.

Example 6.2.2 *Phase Plane Near Equilibria*

Determine the phase plane of example (3a)–(3b),

$$\frac{dx}{dt} = f(x, y) = -x + xy, \tag{17a}$$

$$\frac{dy}{dt} = g(x, y) = -4y + 8xy, \tag{17b}$$

near the equilibria (0, 0) and $\left(\frac{1}{2}, 1\right)$ found in Example 6.2.1.

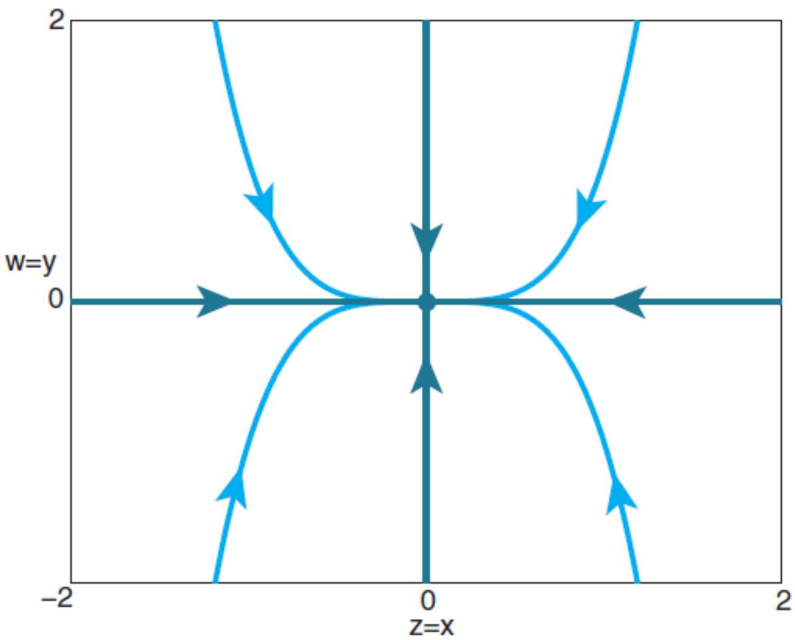

Figure 6.2.1 Stable node at (0, 0).

• SOLUTION. Since

$$\frac{\partial f}{\partial x} = -1 + y,\ \frac{\partial f}{\partial y} = x,\ \frac{\partial g}{\partial x} = 8y,\ \frac{\partial g}{\partial y} = -4 + 8x, \tag{18}$$

the matrix for this example is

$$\begin{bmatrix} -1+y & x \\ 8y & -4+8x \end{bmatrix}. \tag{19}$$

EQUILIBRIUM $(0, 0)$**.** Consider first the equilibrium $x = 0$, $y = 0$. In this case the matrix is $\mathbf{A} = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -4 \end{smallmatrix}\right]$. By Theorem 6.2.1, the behavior of (17a)–(17b) near (0, 0) is that of (9a)–(9b), which is

$$\frac{dz}{dt} = -z, \tag{20a}$$

$$\frac{dw}{dt} = -4w, \tag{20b}$$

where here $z = x - r = x$ and $w = y - s = y$. In this example, the linear system of differential equations decouples into two first-order differential equations, so that from our study in Chapter 1 of first-order equations (with constant coefficients) it is seen that eigenvalues (roots of the characteristic equation) are -1, -4. Thus, (0, 0) is a stable node as studied in Chapter 4. The eigenvalues λ and eigenvectors (u, v) satisfy the linear system

$$\begin{bmatrix} -1-\lambda & 0 \\ 0 & -4-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{21}$$

The determinant condition is $\det \left[\begin{smallmatrix} -1-\lambda & 0 \\ 0 & -4-\lambda \end{smallmatrix}\right] = 0$. Thus, the characteristic equation is $(\lambda + 1)(\lambda + 4) = 0$ with roots (eigenvalues) $\lambda = -1, -4$ (as already determined),

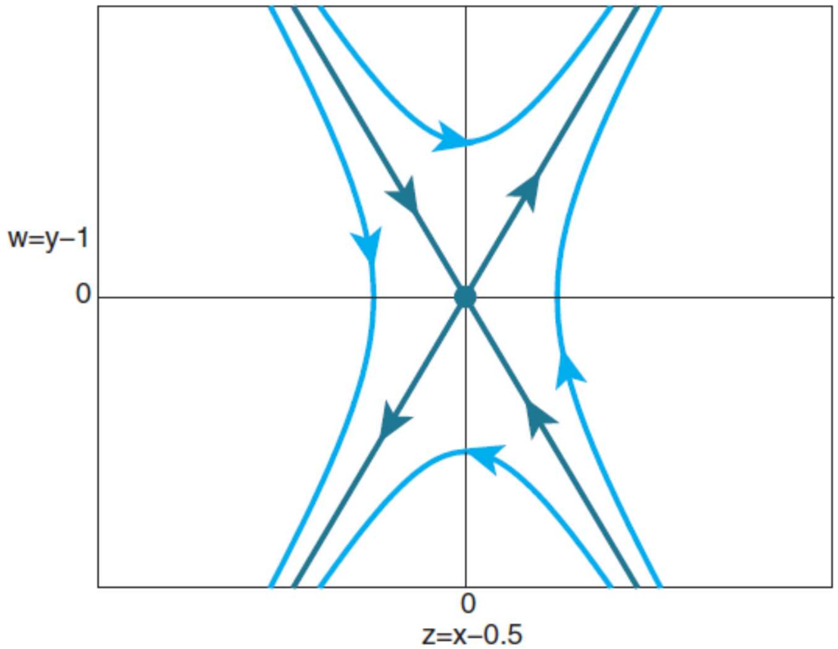

Figure 6.2.2 Saddle point at $\left(\frac{1}{2}, 1\right)$.

so that the equilibrium (0, 0) is a stable node. From (21), for $\lambda = -1$, since $-3v = 0$, the corresponding eigenvector is any multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the x-axis, while for $\lambda = -4$, the corresponding eigenvector is any multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, since $3u = 0$ is in the direction of the y-axis. Near (0, 0) the phase plane of the nonlinear system (17a)–(17b) will resemble that of its linearization, (20a)–(20b), which is shown in figure 6.2.1. ♦

EQUILIBRIUM $\left(\frac{1}{2}, 1\right)$. For the equilibrium $x = \frac{1}{2}$, $y = 1$, the linearization involves the displacement from the equilibrium, so that here $z = x - r = x - \frac{1}{2}$ and $w = y - s = y - 1$. For this equilibrium, $(\frac{1}{2}, 1)$, the matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} \\ 8 & 0 \end{bmatrix}. \tag{22}$$

The behavior of the nonlinear system (17a)–(17b)near the equilibrium $(\frac{1}{2}, 1)$ can be approximated by its linearized system

$$\frac{dz}{dt} = \frac{1}{2}w, \tag{23a}$$

$$\frac{dw}{dt} = 8z. \tag{23b}$$

The eigenvalues λ and eigenvectors (u, v) satisfy the linear system

$$\begin{bmatrix} -\lambda & \frac{1}{2} \\ 8 & -\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{24}$$

The determinant condition is

$$\det \begin{bmatrix} -\lambda & \frac{1}{2} \\ 8 & -\lambda \end{bmatrix} = 0. \tag{25}$$

Thus, the characteristic equation is $\lambda^2 - 4 = 0$ with roots (eigenvalues) $\lambda = \pm 2$, so that the equilibrium $\left(\frac{1}{2}, 1\right)$ is a saddle point. From (24), for $\lambda = 2$, since $-2u + \frac{1}{2}v = 0$, the

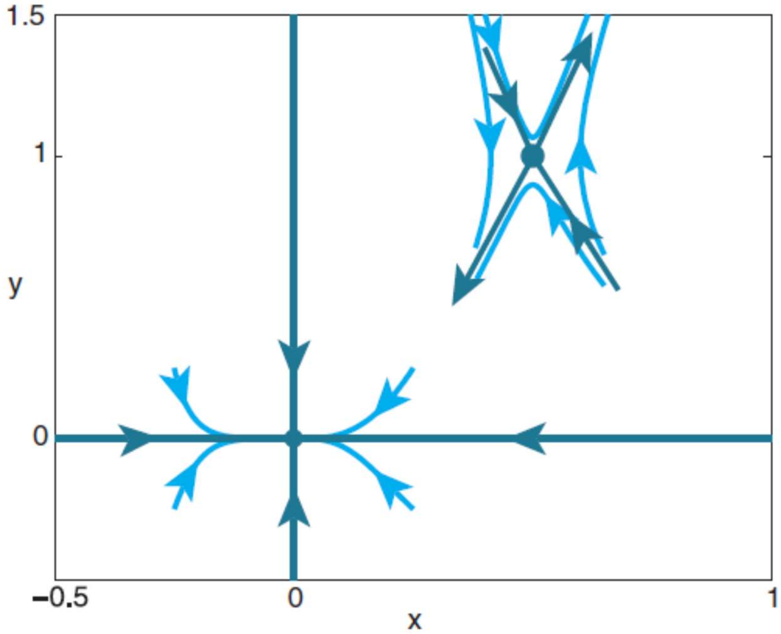

Figure 6.2.3 Phase plane using only linearizations.

corresponding eigenvector is any multiple of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, while for $\lambda = -2$, the corresponding eigenvector is any multiple of $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ since $2u + \frac{1}{2}v = 0$. Thus, the local phase plane is shown in figure 6.2.2, with the corresponding eigenvectors being the stable and unstable directions for the saddle point $\left(\frac{1}{2}, 1\right)$.

PHASE PLANE OF A NONLINEAR SYSTEM. It is important to keep in mind that Theorem 6.2.1 describes only the behavior near each equilibrium, as shown in figure 6.2.3.

6.2.2 Nonlinear Systems: Summary, Philosophy, Phase Plane, Direction Field, Nullclines

Let us summarize the method for analyzing nonlinear systems:

1. Find equilibrium.
2. Linearize in the neighborhood of each equilibrium.
3. For each linearization, find eigenvalues. Find eigenvectors for the real eigenvalues.
4. For the usual cases described above, sketch the phase plane for the nonlinear system near each equilibrium by using the phase plane of the corresponding linear system.
5. Determine trajectories in the phase plane away from equilibrium by using
   a) direction field (or numerical solutions) from graphics software, described below;
   b) optional method of nullclines, described shortly, to understand 5(a).

In this subsection, we will do some examples to explain these ideas. More examples will be done in other sections of this chapter.

Direction Field

Here we will present some basic results for nonlinear systems of differential equations of the form

$$\frac{dx}{dt} = f(x, y), \tag{26a}$$

$$\frac{dy}{dt} = g(x, y). \tag{26b}$$

One of the beauties of mathematics is that we can determine the **direction field** for the phase plane directly from the differential equation. The direction field is a vector field in the tangential direction defined by the differential equation $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (f(x, y), g(x, y))$. It is not necessary to determine the phase plane from brute force numerical solutions of the differential equation. Sometimes the following observation is useful.

For solutions that are not equilibria, solutions in the phase plane will be curves. Trajectories or orbits of the solution will move in time. From the differential equation (26a), we see that if $f(x, y) > 0$ then $\frac{dx}{dt} > 0$, which means x increases in time, and we introduce arrows moving toward the right ($\rightarrow$) in the x, y-plane. Similarly, if $f(x, y) < 0$ then x decreases in time, and we introduce arrows moving to the left ($\leftarrow$). From the other differential equation, (26b),we see that if $g(x, y) > 0$ then $\frac{dy}{dt} > 0$, which means y increases in time, and we introduce arrows moving upward ($\uparrow$) in the x, y-plane. And finally, if $g(x, y) < 0$, then $\frac{dy}{dt} < 0$, and we introduce downward arrows ($\downarrow$) in the x, y-plane. This is similar to drawing lines indicating which direction (northeast or southeast) the wind is going.

By using the chain rule, we obtain from the system (26a)–(26b),

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x, y)}{f(x, y)}, \tag{27}$$

from which we may determine the phase plane. Equation (27) is a first-order differential equation, and we showed in Chapter 1 how easy it is for computers to sketch its slope field. It is somewhat subtle that the direction field is the slope field supplemented by the vector direction.

Example 6.2.3 *Direction Field and Phase Plane*

Find the direction field and phase plane for system (17a)–(17b),

$$\frac{dx}{dt} = -x + xy, \tag{28a}$$

$$\frac{dy}{dt} = -4y + 8xy. \tag{28b}$$

• SOLUTION. The direction field is shown in figure 6.2.4. Arrows have been included to make it a direction field. The phase plane for this example is shown in figure 6.2.5. It is obtained by combining information from the direction field with our knowledge of each equilibrium. The equilibrium $(0, 0)$ is a stable node with eigenvalue -1 with corresponding eigenvector $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, and with eigenvalue -4 with corresponding

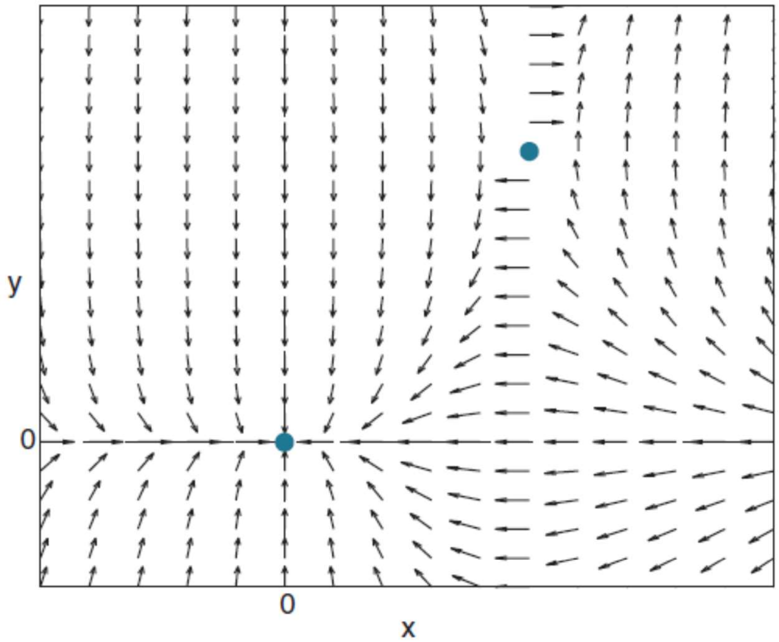

Figure 6.2.4 Direction field for (28a)–(28b).

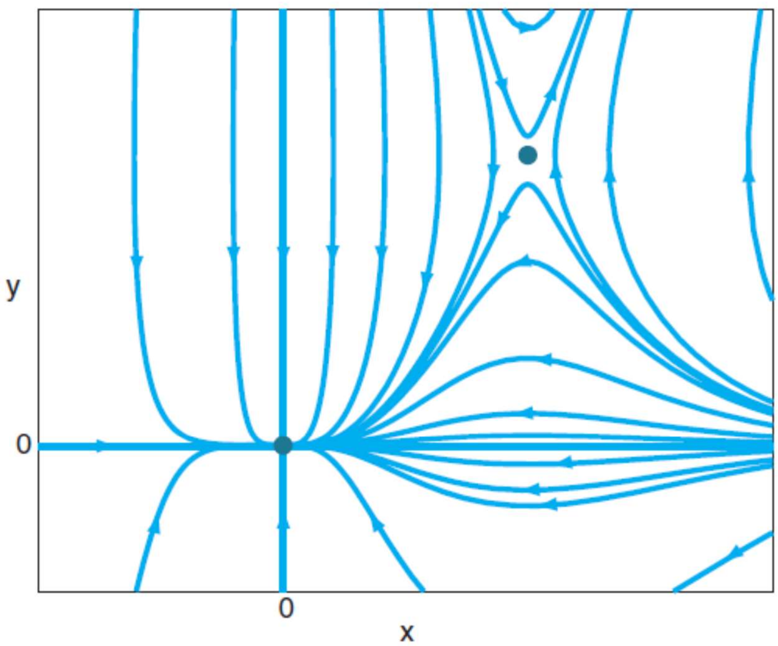

Figure 6.2.5 Phase plane for (28a)–(28b).

eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The other equilibrium $\left(\frac{1}{2}, 1\right)$ is an (unstable) saddle point with eigenvalue 2 with corresponding unstable eigenvector direction $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and with eigenvalue -2 with corresponding stable eigenvector direction $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

Method of Nullclines

It is easy for computers to sketch the phase plane (with or without the direction field) of a given nonlinear system of differential equations,

$$\frac{dx}{dt} = f(x, y), \tag{29a}$$

$$\frac{dy}{dt} = g(x, y). \tag{29b}$$

However, the method of nullclines can be used to understand why solutions behave the way they appear. The phase plane satisfies the slope field of the nonlinear first-order differential equation obtained by dividing the two equations to give

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}. \tag{30}$$

First graph $f(x, y) = 0$. Along this curve from (30), $\frac{dy}{dx} = \infty$, so we draw short vertical dashes (see figure 6.2.6) since the trajectories have vertical tangents there. Similarly, along $g(x, y) = 0$, $\frac{dy}{dx} = 0$, so we draw short horizontal dashes. Both curves are called **nullclines**. Interestingly enough, the intersection of nullclines will provide a graphical determination of the equilibria. More important, we note that in the region in which $f(x, y) > 0$, $\frac{dx}{dt} > 0$, so that x increases as a function of time which we can mark in the phase plane with a right arrow $\rightarrow$ (and vice versa, $f(x, y) < 0$ corresponds to x decreasing in time, $\leftarrow$). Similarly, regions with $g(x, y) > 0$ correspond to y increasing in time, which we indicate in the phase plane with an upper arrow $\uparrow$ (and vice versa meaning $g(x, y) < 0$ corresponds to y decreasing in time, $\downarrow$). We combine the arrows, as shown in the following example, so that we can predict and mark the direction of the flow in the phase plane directly from the differential equation.

In Exercises 1–12,

(a) Determine all equilibria and classify (node, saddle, spiral, center, stable or unstable).
(b) If an eigenvalue is real, find the eigenvectors.
(c) Graph the phase plane using the phase plane of linearized system.
(d) Graph the phase plane using the direction field from software.
(e) Graph the phase plane using the method of nullclines and part (c).

1. $\frac{dx}{dt} = x + xy,\ \frac{dy}{dt} = 2y - 4xy.$

2. $\frac{dx}{dt} = -x + xy,\ \frac{dy}{dt} = -2y + 8xy.$

3. $\frac{dx}{dt} = 2x - 2xy,\ \frac{dy}{dt} = y - xy.$

4. $\frac{dx}{dt} = 1 - x^2,\ \frac{dy}{dt} = y + 1.$

5. $\frac{dx}{dt} = x - y,\ \frac{dy}{dt} = -2x + 2xy.$

6. $\frac{dx}{dt} = y^3 + 1,\ \frac{dy}{dt} = x^2 + y.$

7. $\frac{dx}{dt} = 1 - y^2,\ \frac{dy}{dt} = 1 - x^2.$

8. $\frac{dx}{dt} = x(1 - y^2),\ \frac{dy}{dt} = x + y.$

9. $\frac{dx}{dt} = x - y + x^2,\ \frac{dy}{dt} = x + y.$

10. $\frac{dx}{dt} = 2x - y - xy,\ \frac{dy}{dt} = x + 2y.$

11. $\frac{dx}{dt} = -x - 2y,\ \frac{dy}{dt} = 2x - y + xy^2.$

12. $\frac{dx}{dt} = 2x + y,\ \frac{dy}{dt} = -x - 2y + y^3.$

Exercises 13–17 refer to $\frac{dx}{dt} = x - xy + \gamma x^2$, $\frac{dy}{dt} = -y + xy$. In each case find all equilibria and classify (node, saddle, spiral, center, stable or unstable).

13. $\gamma = -8$.

14. $\gamma = -\dfrac{1}{3}$.

15. $\gamma = \dfrac{1}{3}$.

16. $\gamma = 1$.

17. $\gamma = 8$.