محاضرة تمهيدى ماجستير

مناقشة الاتى وحل بعض التمارين من كتاب

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10

Harmonic Functions

In Chapter 5, we saw that if an analytic function has a continuous second derivative, then the real (or imaginary) part of the function is harmonic. In Chapter 8, it was shown that *all* analytic functions are infinitely differentiable and in particular, have continuous second derivatives. Thus, the real part of an analytic function is always harmonic.

In this chapter, we examine the extent to which the converse is true. In simply connected domains, we show that every harmonic function is the real part of *some* analytic function. This result enables us to prove several theorems for harmonic functions that are analogous to theorems for analytic functions. In particular, an analog to Cauchy's integral formula, known as Poisson's integral formula, gives a method for determining the values of a harmonic function inside a disk from the behavior at its boundary points.

10.1 Comparison with Analytic Functions

Recall that a continuous real-valued function $u(x, y)$, defined and single-valued in a domain D, is said to be harmonic in D if it has continuous first and second partial derivatives that satisfy Laplace's equation

$$u_{xx} + u_{yy} = 0.$$

In Section 5.3, we illustrated how the Cauchy–Riemann equations might be used to construct a function $v(x, y)$ conjugate to a given harmonic function $u(x, y)$; that is, a function $v(x, y)$ was found for which $f(z) = u(x, y) + iv(x, y) = u(z) + iv(z)$ was analytic. The method entailed finding all functions $v(z)$ satisfying the two conditions

$$u_x = v_y, \quad u_y = -v_x.$$

This method was successful when the partial integration $\int v_y \, dy$ could explicitly be solved. We now give general conditions for the existence of an analytic function whose real part is a prescribed harmonic function. First note that in view of the Cauchy–Riemann equations, the derivative of any analytic function $f(z) = u(z) + iv(z)$ may be expressed as

$$f'(z) = u_x(z) - iu_y(z).$$

Hence we can find f (by integration) directly from u. The details follow.

Theorem 10.1. *If u is harmonic on a simply connected domain D, then there exists an analytic function on D whose real part equals u.*

Proof. Set $g(z) = u_x(z) - iu_y(z) := U(z) + iV(z)$, $z \in D$. Then by Laplace's equation,

$$U_x - V_y = u_{xx} - (-u_{yy}) = 0. \tag{10.1}$$

Since the mixed partial derivatives of $u(z)$ are continuous in D,

$$U_y + V_x = (u_x)_y + (-u_y)_x = 0. \tag{10.2}$$

But (10.1) and (10.2) are the Cauchy–Riemann equations for $g = U + iV$. Noting that U_x, U_y, V_x, V_y are all continuous, we may apply Theorem 5.17 to establish the analyticity of $g(z)$ in D.

Next choose any point z_0 in D, and set

$$F(z) = \int_{z_0}^{z} g(\zeta)\, d\zeta.$$

Then, by Corollary 8.15, $F(z)$ is analytic in D with

$$F'(z) = g(z) = u_x(z) - iu_y(z).$$

Observe that the derivative of $F(z)$ may also be expressed as

$$F'(z) = \frac{\partial}{\partial x}\operatorname{Re} F(z) - i\frac{\partial}{\partial y}\operatorname{Re} F(z).$$

Thus $u(z)$ and $\operatorname{Re} F(z)$ have the same first partial derivatives in D, so that

$$\operatorname{Re} F(z) = u(z) + c \quad (c \text{ a real constant}).$$

Hence, the function

$$f(z) = F(z) - c = \int_{z_0}^{z} (u_x(\zeta) - iu_y(\zeta))\, d\zeta - c$$

is analytic in D with $\operatorname{Re} f(z) = u(z)$. ■

Corollary 10.2. *If u is harmonic on a simply connected domain D, then there exists an analytic function on D whose imaginary part equals u.*

Proof. By Theorem 10.1, there exists an analytic function $h(z)$ such that $\operatorname{Re} h(z) = u(z)$. But then $f(z) = ih(z)$ is analytic with $\operatorname{Im} f(z) = \operatorname{Re} h(z) = u(z)$. ■

Example 10.3. Let $u(x, y) = \sin x \cosh y + \cos x \sinh y + x^2 - y^2 + 2xy$. It can be easily seen that u is harmonic in $\mathbb{C}$. Following the proof of Theorem 10.1,

$$\begin{aligned} f'(z) = u_x - iu_y &= \cos x \cosh y - \sin x \sinh y + 2x + 2y \\ &\qquad - i\,(\sin x \sinh y + \cos x \cosh y - 2y + 2x). \end{aligned}$$

As $\cos(iy) = \cosh y$ and $-i\sin(iy) = \sinh y$, we can simplify the last equation and obtain

$$f'(z) = (1 - i)(\cos z + 2z).$$

Thus, $f(z) = (1 - i)(\sin z + z^2) + c$. ●

The requirement in Theorem 10.1 that the domain be simply connected is essential. For example, the function

$$u(z) = u(x, y) = \ln \sqrt{x^2 + y^2} = \ln |z|$$

is harmonic in the punctured plane $\mathbb{C}\setminus\{0\}$. Each point in $\mathbb{C}\setminus\{0\}$ has a neighborhood where $\log z$ has a single-valued analytic branch. In other words, we say that $u(z)$ is locally the real part of an analytic function as guaranteed by Theorem 10.1. Therefore, $u(z) = \ln |z|$, being the real part of an analytic function, is harmonic in such neighborhoods. We also know that the principal logarithm $\operatorname{Log} z$ defined by

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$$

is analytic in the cut plane $D = \mathbb{C}\setminus(-\infty, 0]$. Now if some function

$$f(z) = \ln |z| + iv(z)$$

were analytic throughout the punctured plane $\mathbb{C}\setminus\{0\}$, then g defined by

$$g(z) = f(z) - \operatorname{Log} z$$

would be analytic in the slit plane $D = \mathbb{C}\setminus(-\infty, 0]$. Since $g(z)$ is purely imaginary in D, an application of the Cauchy–Riemann equations shows that $g(z)$ must be constant in D. Thus, any function analytic in D whose real part is $\ln |z|$ must be of the form

$$u(z) + iv(z) = \operatorname{Log} z + ic,$$

where c is a real constant. It follows that $v(z) = \operatorname{Arg} z + c$. But then

$$\lim_{\substack{y \to 0 \\ y > 0}} v(-1 + iy) = \lim_{\substack{y \to 0 \\ y > 0}} \operatorname{Arg}(-1 + iy) + c = \pi + c$$

and

$$\lim_{\substack{y\to 0\\ y<0}} v(-1+iy) = \lim_{\substack{y\to 0\\ y<0}} \operatorname{Arg}(-1+iy) + c = -\pi + c$$

which means that v is discontinuous at -1, a contradiction. An argument similar to this shows that v is not continuous at all points in the negative real axis $(-\infty, 0]$. Thus, there is no hope for defining an analytic function in $\mathbb{C}\setminus\{0\}$ whose real part is $u(z) = \ln|z|$. Hence, *a harmonic function need not have an analytic completion in a multiply connected domain.*

In view of Theorem 10.1, we may now modify some of the theorems in Chapter 8 to obtain harmonic analogs. Our next theorem is the harmonic analog of Liouville's theorem.

Theorem 10.4. *A function harmonic and bounded in $\mathbb{C}$ must be a constant.*

Proof. Suppose $u(z)$ is harmonic and bounded in the plane. Theorem 10.1 guarantees the existence of an entire function $f(z)$ whose real part is $u(z)$. But then

$$g(z) = e^{f(z)}$$

is an entire function too. Since $|g(z)| = e^{u(z)}$, $g(z)$ is also bounded in the plane. By Liouville's theorem $g(z)$, and hence $u(z) = \ln|g(z)|$, must be constant. ■

Clearly, Theorem 10.4 may be restated in a general form as follows:

Theorem 10.5. *If the real or imaginary part of an entire function is bounded above or below by a real number M, then the function is a constant.*

We now prove an analog to Gauss's mean-value theorem for analytic functions. This is one of the fundamental facts about harmonic functions, called the mean value property of harmonic functions.

Theorem 10.6. (Mean Value Property) *Suppose $u(z)$ is harmonic in a domain containing the disk $|z - z_0| \leq R$. Then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta.$$

Proof. Let $f(z)$ be a function analytic in $|z - z_0| \leq R$ whose real part is $u(z)$. By Gauss's mean-value theorem,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta.$$

The result follows upon taking real parts of both sides. ■

The right-hand side of the last formula gives in particular that the mean (or average) value u on the circle $|z - z_0| = R$ is simply the value of u at the center of the circle $|z - z_0| = R$. In Section 10.2, we shall consider a similar expression for a point of the disk $|z - z_0| < R$ other than the center. We have shown that the behavior of a harmonic function on the boundary of a closed and bounded region determines the behavior of the harmonic function throughout the region. For instance, a harmonic function u in the unit disk $|z| < 1$ that extends continuously to $|z| \leq 1$ is completely determined by its values on the boundary $|z| = 1$. The explicit formula for the value of u for each point in $|z| < 1$ is given by the Poisson integral formula for a harmonic function and this is the subject of the discussion in Section 10.2. Unlike the situation for analytic functions, this result cannot be improved to an arbitrary sequence of points in the region. For instance, the nonconstant function $u(z) = x$ is harmonic in the plane with $u(z) \equiv 0$ on the imaginary axis. Hence, "analytic" cannot be replaced with "harmonic" in the statement of Theorem 8.47. That is, even if $u(z)$ is harmonic in a domain D, $u(z_n) \equiv 0$, and $z_n \to z_0$ in D, we are not guaranteed that $u(z) \equiv 0$ in D. Thus, the analog of the identity principle (see Theorem 8.48) for analytic functions does not hold for harmonic functions. However, we can salvage the following:

Theorem 10.7. *If $u(z)$ is harmonic in a domain D and constant in the neighborhood of some point in D, then $u(z)$ is constant throughout D.*

Proof. Let A be the set of all points z_0 in D for which $u(z)$ is constant in some neighborhood of z_0. Clearly A is a nonempty open set. To prove that $A = D$, it suffices to show that $B = D \backslash A$ is open, for then B would have to be empty in order for D to be connected.

Suppose B is not open. Then for a point z_0 in B and an $\epsilon > 0$ there is a point z_1 in A such that $z_1 \in N(z_0; \epsilon) \subset D$. Since A is open, we can find a $\delta > 0$ sufficiently small so that $N(z_1; \delta) \subset N(z_0; \epsilon) \cap A$. Now construct an analytic function $f(z)$ such that

$$\operatorname{Re} f(z) = u(z) \quad \text{for all } z \text{ in } N(z_0; \epsilon).$$

Since $u(z)$ is constant in $N(z_1; \delta)$, $f'(z) = 0$ for z in $N(z_1; \delta)$. An application of Theorem 8.47 to $f'(z)$ shows that $f'(z) \equiv 0$ throughout $N(z_0; \epsilon)$. Then, by Theorem 5.9, $f(z)$ is constant in $N(z_o; \epsilon)$. Hence, $u(z) = \operatorname{Re} f(z)$ is also constant in $N(z_0; \epsilon)$, contradicting the assumption that $z_0 \in B$. ∎

Example 10.8. Suppose that $u(z)$ is harmonic in a domain D such that the set $\{z \in D : u_x(z) = 0 = u_y(z)\}$ has a limit point in D. Then we can easily show that $u(z)$ is a constant throughout D.

To see this, we define

$$F(z) = u_x(z) - iu_y(z), \quad z \in D.$$

Then F is analytic in D and the set $\{z \in D : F(z) = 0\}$ has a limit point in D. By the uniqueness theorem for analytic functions (see Theorem 8.47), $F(z) \equiv 0$ in D and so, $u_x(z) = 0 = u_y(z)$ on D, i.e., u is a constant. ∎

Analogous to the maximum and minimum modulus theorems for analytic functions are the maximum and minimum principles for harmonic functions. The fact that a harmonic function is locally the real part of an analytic function produces a number of important results. One of them is the maximum principle.

Theorem 10.9. (Maximum Principle for Harmonic Functions) *A nonconstant harmonic function cannot attain a maximum or a minimum in a domain.*

Note that a harmonic function $u(z)$ attains a maximum at a point z_0 if and only if the harmonic function $-u(z)$ attains a minimum at z_0. So the minimum principle can be derived directly from the maximum principle. This result has several proofs.

Proof. The maximum modulus theorem for analytic functions is a direct consequence of Gauss's mean-value theorem and the fact that an analytic function is continuous. Similarly, we may deduce the maximum principle for harmonic functions from the mean-value principle for harmonic functions (Theorem 10.6). Indeed, we assume that $u(z)$ attains the maximum at $z_0 \in D$. Then, for each r with $0 < r \leq \text{dist}\,(z_0, D)$, Theorem 10.6 gives

$$\frac{1}{2\pi}\int_0^{2\pi} (u(z_0) - u(z_0 + Re^{i\theta}))\,d\theta = 0.$$

Since $u(z_0) - u(z_0 + Re^{i\theta})$ is a continuous function of θ and is nonnegative, we have

$$u(z_0) = u(z_0 + Re^{i\theta}) \quad \text{for } 0 \leq \theta \leq 2\pi.$$

Thus, $u(z) = u(z_0)$ for all z in some neighborhood $N(z_0; \delta)$. Hence, $u(z) = u(z_0)$ on D (see Theorem 10.7).

For a second proof, we assume that $u(z)$ is a nonconstant function harmonic in a domain D. Given z_0 in D, construct a function $f(z) = u(z) + iv(z)$ that is analytic in some neighborhood $N(z_0; \delta)$ of z_0.

We set $g(z) = e^{f(z)}$, and note that $|g(z)| = e^{u(z)}$. If z_0 were a maximum for $u(z)$ in this neighborhood, then z_0 would be a maximum for $|g(z)|$. By the maximum modulus theorem for analytic functions, the function g must be constant on $N(z_0; \delta)$. Therefore, u is constant on $N(z_0; \delta)$ and hence on D, which contradicts the assumption that u is nonconstant. The proof is complete.

Alternatively, one could use the open mapping theorem (Theorem 9.55). Then it follows that there exists an $\epsilon > 0$ such that $N(f(z_0); \epsilon)$ is contained in the image of $N(z_0; \delta)$ under $f(z)$. In particular, for some point $z_1 \in N(z_0; \delta)$ we have $\text{Re}\, f(z_1) = u(z_0) + \epsilon/2$. Thus, z_0 is not a maximum of $u(z)$ in D. ∎

Observe that $\min\{|f(z)| : z \in \overline{D}\}$ may be attained at an interior point of D without the analytic function f on $\overline{D}$ being constant. For example, consider $f(z) = z$, for $|z| < 1$. Then, for $|z| \leq r$ $(r < 1)$,

$$|f(z)| = |z| \geq 0 = |f(0)|$$

so that the minimum modulus of $f(z)$ is attained at the interior point $z = 0$. However, the maximum of $|f(z)|$ on $|z| \leq r$ is attained at $z = r$ which is a boundary point of $|z| < r$.

The minimum principle for harmonic functions is actually stronger than the minimum modulus theorem for analytic functions. The hypothesis that the function be nonzero in the domain is unnecessary for harmonic functions. Of course, a harmonic function can assume negative values in a domain, whereas the modulus of an analytic function cannot.

Corollary 10.10. *Suppose $u(z)$ is harmonic in a bounded domain D whose boundary is the closed contour C. If $u(z)$ is continuous in $D \cup C$, with $u(z) \equiv K (K$ a constant) on C, then $u(z) \equiv K$ throughout D.*

Proof. Since $D \cup C$ forms a compact set, $u(z)$ must attain a maximum and minimum. By Theorem 10.9, the maximum and minimum cannot occur in D. Thus, they must occur on C. But this means that $\max u(z) = \min u(z) = K$. Hence, $u(z) \equiv K$ throughout D. ∎

The boundedness of D in Corollary 10.10 is essential. The domain $\{z : \operatorname{Re} z > 0\}$ has the boundary $\{z : \operatorname{Re} z = 0\}$. The function $u(z) = x$ is continuous for $\operatorname{Re} z \geq 0$ with $u(z) \equiv 0$ on the boundary. But $u(z) \neq 0$ for $\operatorname{Re} z > 0$.

Corollary 10.11. *Suppose $u_1(z)$ and $u_2(z)$ are harmonic in a bounded domain D whose boundary is the closed contour C. If $u_1(z)$ and $u_2(z)$ are continuous in $D \cup C$, with $u_1(z) \equiv u_2(z)$ on C, then $u_1(z) \equiv u_2(z)$ throughout D.*

Proof. Set $u(z) = u_1(z) - u_2(z)$ and apply Corollary 10.10. ∎

Example 10.12. Suppose that $f(z)$ is an entire function such that $f(z)$ is real on the unit circle $|z| = 1$. Then $f(z)$ is constant.

To see this, we set $f = u + iv$. By assumption, $v(z) = 0$ on $|z| = 1$. By Corollary 10.10, $v(z) = 0$ for $|z| < 1$. Hence, $f(z)$ is real for $|z| < 1$, i.e., $f(|z| < 1) \subseteq \mathbb{R}$. By the open mapping theorem, f must be constant for $|z| < 1$. By the uniqueness theorem for analytic functions, f must be a constant throughout $\mathbb{C}$. ∎

There is an interesting relationship between the maximum modulus of an analytic function and the maximum of its real part.

Theorem 10.13. (Borel–Carathéodory) *Suppose $f(z)$ is analytic in the disk $|z| \leq R$. Let $M(r) = \max_{|z|=r} |f(z)|$ and $A(r) = \max_{|z|=r} \operatorname{Re} f(z)$. Then for $0 < r < R$,*

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|.$$

Proof. If $f(z)$ is constant (say $f(z) = k$), then the right-hand side is bounded below by

$$\frac{-2r}{R-r}|k| + \frac{R+r}{R-r}|k| = |k| = M(r),$$

and the result follows. Hence, we may assume that $f(z)$ is nonconstant.

If $f(0) = 0$, then by Theorem 10.9, $A(R) > A(0) = 0$. Since

$$\operatorname{Re}\{2A(R) - f(z)\} \geq A(R) > 0$$

for $|z| \leq R$, and

$$|2A(R) - f(z)|^2 \geq |f(z)|^2 + 4A(R)[A(R) - \operatorname{Re} f(z)] \geq |f(z)|^2,$$

the function

$$g(z) = \frac{f(z)}{2A(R) - f(z)}$$

is analytic and $|g(z)| \leq 1$ for $|z| \leq R$. Then by Schwarz's lemma,

$$\max_{|z|=r} |g(z)| \leq r/R.$$

But

$$|f(z)| = \left|\frac{2A(R)g(z)}{1+g(z)}\right| \leq \frac{2A(R)r/R}{1-r/R} = \frac{2rA(R)}{R-r}, \tag{10.3}$$

and the result follows when $f(0) = 0$.

Finally, if $f(0) \neq 0$, we apply (10.3) to $f(z) - f(0)$. This leads to

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \max_{|z|=r} \operatorname{Re}\{f(z) - f(0)\} \leq \frac{2r}{R-r}(A(R) + |f(0)|).$$

Thus

$$|f(z)| \leq \frac{2r}{R-r}(A(R) + |f(0)|) + |f(0)| = \frac{2r}{R-r}A(R) + \frac{R+r}{R-r}|f(0)|,$$

and the theorem is proved. ∎

Theorem 10.13 may be used to generalize both Theorem 8.35 and Theorem 10.4 as follows.

Theorem 10.14. *Suppose $f(z)$ is an entire function and that* $\operatorname{Re} f(z) \leq Mr^\lambda$ *for $|z| = r \geq r_0$ and for some nonnegative real number λ. Then $f(z)$ is a polynomial of degree at most $[\lambda]$.*

Proof. Set $R = 2r$ in Theorem 10.13. Then

$$|f(z)| \leq \frac{2r}{2r-r}A(2r) + \frac{2r+r}{2r-r}|f(0)| \leq 2(2r)^\lambda M + 3|f(0)| \leq M_1 r^\lambda,$$

for M_1 sufficiently large. The result now follows from Theorem 8.35. ∎

Questions 10.15.

1. When can we say $\ln|f(z)|$ is harmonic? Where is it harmonic?
2. In the proof of Theorem 10.1, where did we use the fact that the domain was simply connected?
3. What theorems are valid for disks but not for a simply connected domain?
4. Where was continuity of the second partial derivatives for harmonic functions important?
5. Can a nonconstant function harmonic in the plane omit more than one real value?
6. Let $f = u + iv$ be analytic in a domain D. Is u_{xx} harmonic in D?
7. Can the maximum modulus theorem for analytic functions be proved using the maximum principle for harmonic functions?
8. Suppose a function is harmonic in a domain D and continuous on its boundary C. Must the function be continuous in $D \cup C$?
9. For a harmonic function u in a domain D which vanishes in an open subset of D, does u vanish identically in D?
10. Is there a relationship between the coefficients of an analytic function and the maximum of its real part?
11. Why is Theorem 10.14 a generalization of Theorem 8.35 and Theorem 10.4?
12. Is every harmonic function an open mapping?
13. Let Ω be a domain and $u \in C^3(\Omega)$. If u is harmonic on Ω, must u_x be harmonic on Ω? Must u_y be harmonic on Ω?
    Note: $C^k(\Omega)$ denotes the set of all functions u whose partial derivatives of order k all exist and are continuous on Ω.
14. What is the average value of the harmonic function $u(x, y) = xy$ on the circle $(x - 2)^2 + (y + 1)^2 = 1$?
15. Let $u(z)$ be harmonic on the disk $|z| < r$ such that $u_x(z) = 0$ on $|z| < r$. What can we conclude about u?
16. Let u be harmonic for $|z| < 1$. Suppose that $\{z_n\}_{n\geq 1}$ is a sequence of complex numbers not equal to z_0 such that $z_n \to z_0$ in $|z| < 1$ and $u(z_n) = 0$ for $n \in \mathbb{N}$. Must u be identically zero? If not, under what additional assumption, do we get $u \equiv 0$?
17. Must a product of two harmonic functions u and v be harmonic?
18. Suppose that u is harmonic in a domain D and v is its harmonic conjugate. Must uv be harmonic on D? Must u^2 be harmonic on D?
19. We know that $u(z) = \ln|z|$ is harmonic in the annulus $D = \{z : 1 < |z| < 2\}$. Can $u(z)$ have a harmonic conjugate on D?

Exercises 10.16.

1. Show that a function harmonic in a domain must have partial derivatives of all orders.

2. If $u(z)$ is nonconstant and harmonic in the plane, show that $u(z)$ comes arbitrarily close to every real value.
3. Prove the minimum principle directly by each of the three methods in which the maximum principle was proved.
4. Show that $\int_0^\pi \ln \sin \theta \, d\theta = -\pi \ln 2$ by applying the mean-value principle to $\ln|1+z|$ for $|z| \le r < 1$, and then letting $r \to 1$.
5. Suppose $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour C, with $\operatorname{Re} f(z) = \operatorname{Re} g(z)$ on C. Show that $f(z) = g(z) + i\beta$ inside C, where β is a real constant.
6. Generalize the previous exercise by showing that the conclusion still holds if it is only assumed that $f(z)$ and $g(z)$ are analytic inside C and continuous in the region consisting of C and its interior.
7. If $u(z)$ is harmonic and bounded in the punctured disk $0 < |z - z_0| < R$, show that $\lim_{z \to z_0} u(z)$ exists.
8. Suppose $u_1(z)$ and $u_2(z)$ are harmonic in a simply connected domain D, with $u_1(z)u_2(z) \equiv 0$ in D. Prove that either $u_1(z) \equiv 0$ or $u_2(z) \equiv 0$ in D.
9. It is easy to see that $u(z) = \operatorname{Im} \left(\frac{1+z}{1-z}\right)^2$ is harmonic in the unit disk $|z| < 1$ and $\lim_{r \to 1^-} u(re^{i\theta}) = 0$ for all θ. Why does this not contradict the maximum principle for harmonic functions? Is u continuous on $|z| = 1$?
10. Does there exist a harmonic function in $|z| < 1$ taking the value 1 everywhere on $|z| = 1$? Is your solution unique?
11. Does there exist a harmonic function on the strip $\{z : 0 < \operatorname{Re} z < 1\}$ with $u(x, 0) = 0$ and $u(x, 1) = 1$? Is your solution unique?
12. If $u(z) = u(x, y)$ is harmonic in the plane with $u(z) \le |z|^n$ for every z, show that $u(z)$ is a polynomial in the two variables x and y.
13. Suppose that $f(z)$ is analytic in the disk $|z| \le R$, and let $A(r) = \max_{|z|=r} \operatorname{Re} f(z)$. Prove that for $r < R$,

$$\max_{|z|=r} \frac{|f^{(n)}(z)|}{n!} \le \frac{2^{n+2}R}{(R-r)^{n+1}} \{A(r) + |f(0)|\}.$$

10.2 Poisson Integral Formula

In this section, we shall attempt to find a harmonic analog to Cauchy's integral formula. If f is analytic inside and on a simple closed contour C, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta \tag{10.4}$$

at all points z inside C. We would like to find an expression for $\operatorname{Re} f$ at points inside C in terms of the values of $\operatorname{Re} f$ on C. Unfortunately, the expression