محاضرة تمهيدى ماجستير

مناقشة الاتى وحل بعض التمارين من كتاب

2-S. Ponnusamy Herb Silverman Complex Variables with Applications

11

Conformal Mapping and the Riemann Mapping Theorem

Our study of mapping properties in Chapters 3 and 4 was limited because derivatives had not yet been introduced. That remedied, we look anew at some old functions. We shall see that the derivative relates the angle between two curves to the angle between their images. In addition, the derivative will be seen to measure the "distortion" of image curves.

Analytic functions mapping disks and half-planes onto disks and half-planes, disks onto the interior of ellipses, etc., have previously been constructed. The major result of this chapter, known as the Riemann mapping theorem, tells us that there is nearly always an analytic function that maps a given simply connected domain onto another given simply connected domain. This is a very powerful result and is used in a wide range of mathematical settings. Our method of proof relies on normal families, a concept that enables us to extract limit functions from families of functions. Recall how we previously had extracted limit points from sequences of points (Bolzano–Weierstrass theorem).

11.1 Conformal Mappings

Any straight line in the plane that passes through the origin may be parameterized by $\sigma(s) = se^{i\alpha}$, where s traverses the set of real numbers and α is the angle−measured in radians−between the positive real axis and the line. More generally, a straight line passing through the point z_0 and making an angle α with the real axis can be expressed as $\sigma(s) = z_0 + se^{i\alpha}$, s real.

Suppose now that a function f is analytic on a smooth (parameterized) curve $z(t)$, $t \in [a, b]$. Then the image of $z(t)$ under f is also a smooth curve whose derivative is given by $f'(z(t))z'(t)$. A smooth curve is characterized by having a tangent at each point. So, we interpret $z'(t)$ as a vector in the direction of the tangent vector at the point $z(t)$. Our purpose is to compare the inclination of the tangent to the curve at a point with the inclination of the tangent to the image curve at the image of the point.

Let $z_0 = z(t_0)$ be a point on the curve $z = z(t)$. Then the vector $z'(t_0)$ is tangent to the curve at the point z_0 and $\arg z'(t_0)$ is the angle this directed tangent makes with the positive x-axis. Suppose that $w = w(t) = f(z(t))$, with $w_0 = f(z_0)$. For any point z on the curve other than z_0, we have the identity

$$w - w_0 = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0).$$

Thus,

$$\arg(w - w_0) = \arg \frac{f(z) - f(z_0)}{z - z_0} + \arg(z - z_0) \pmod{2\pi}, \tag{11.1}$$

where it is assumed that $f(z) \neq f(z_0)$ so that (11.1) has meaning. Note that $\arg(z - z_0)$ is the angle in the z plane between the x axis and the straight line passing through the points z and z_0, while $\arg(w - w_0)$ is the angle in the w plane between the u axis and the straight line passing through the points w and w_0. Hence as z approaches z_0 along the curve $z(t)$, $\arg(z - z_0)$ approaches a value θ, which is the angle that the tangent to the curve $z(t)$ at z_0 makes with the x axis. Similarly, $\arg(w - w_0)$ approaches a value ϕ, the angle that the tangent to the curve $f(z(t))$ at w_0 makes with the u axis.

Suppose $f'(z_0) \neq 0$ so that $\arg f'(z_0)$ has meaning. Then taking limits in (11.1), we find (mod 2π) that

$$\phi = \arg f'(z_0) + \theta, \quad \text{or} \quad \arg w'(t_0) = \arg f'(z_0) + \arg z'(t_0). \tag{11.2}$$

That is, the difference between the tangent to a curve at a point and the tangent to the image curve at the image of the point depends only on the derivative of the function at the point (see Figure 11.1).

For instance, consider $f(z) = z^2$. Then $f'(z) \neq 0$ on $\mathbb{C}\setminus\{0\}$. Choose $z_0 = 1 + i$. Then $f'(z_0) = 2(1 + i)$ so that

$$\arg f'(z_0) = (\pi/4) + 2k\pi.$$

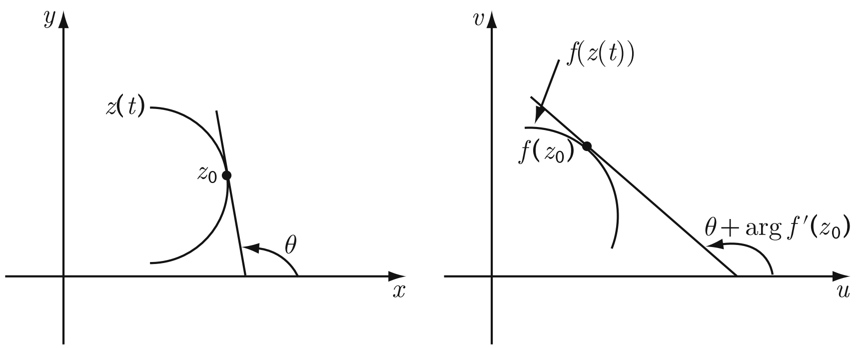

Figure 11.1. The direction of the tangent line at $z(t)$

To verify the angle of rotation of a particular curve, we consider a simple curve C passing through z_0:

$$C : z(t) = t(1+i), \quad t \in \mathbb{R}.$$

Clearly, $\pi/4$ is the angle which the curve C makes with the x axis. The image of C under $f(z) = z^2 = (x^2 - y^2) + i(2xy)$ is given by $w(t) = 0 + 2t^2 i$. Thus, the angle of rotation at $1+i$ is $\pi/2$ which corresponds to the case $k = 0$.

If two smooth curves intersect at a point, then the angle between these two curves is defined as the angle between the tangents to these curves at the point. We can now state

Theorem 11.1. *Suppose $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$. Let $C_1 : z_1(t)$ and $C_2 : z_2(t)$ be smooth curves in the z plane that intersect at $z_0 =: z_1(t_0) =: z_2(t_0)$, with $C_1' : w_1(t)$ and $C_2' : w_2(t)$ the images of C_1 and C_2, respectively. Then the angle between C_1 and C_2 measured from C_1 to C_2 is equal to the angle between C_1' and C_2' measured from C_1' to C_2'.*

Proof. Let the tangents to C_1 and C_2 make angles θ_1 and θ_2, respectively, with the x axis (see Figure 11.2). Then the angle between C_1 and C_2 is $\theta_2 - \theta_1$.

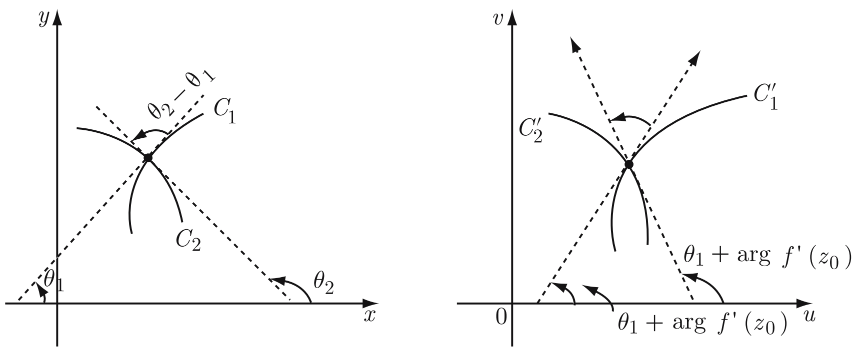

Figure 11.2. The curves C_1 and C_2 intersect at angle α

According to (11.2), the angle between C_1' and C_2', which is the angle between the tangent vectors $f'(z_0)z_1'(t_0)$ and $f'(z_0)z_2'(t_0)$, of the image curves is

$$\theta_2 + \arg f'(z_0) - (\theta_1 + \arg f'(z_0)\ ) = \theta_2 - \theta_1,$$

and the theorem is proved. ■

A function that preserves both angle size and orientation is said to be *conformal*. Theorem 11.1 says that an analytic function is conformal at all points where the derivative is nonzero. We have already discussed a number of examples of conformal maps without referring to the name "conformal".

For instance, $f(z) = e^z$ maps vertical and horizontal lines into circles and orthogonal radial rays, respectively.

A function that preserves angle size but not orientation is said to be *isogonal*. An example of such a function is $f(z) = \overline{z}$. To illustrate, $\overline{z}$ maps the positive real axis and the positive imaginary axis onto the positive real axis and the negative real axis respectively (see Figure 11.3). Although the two curves intersect at right angles in each plane, a "counterclockwise" angle is mapped onto a "clockwise" angle.

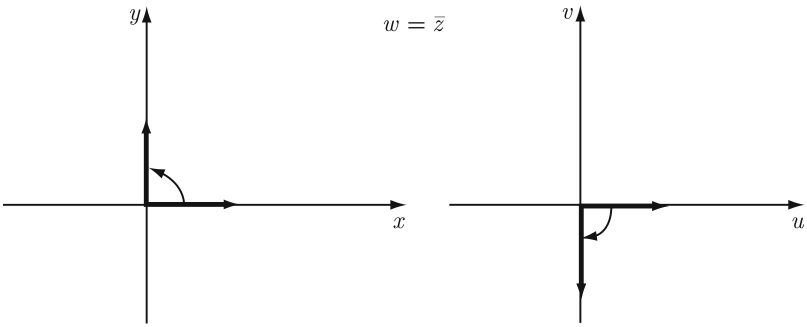

Figure 11.3.

Suppose $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$. When z is near z_0, there is an interesting relationship concerning the distance between the points z and z_0 and the distance between their images. Note that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)(z - z_0)$$

where $\epsilon(z) \to 0$ as $z \to z_0$. Thus for z close to z_0,

$$f(z) \approx f'(z_0)z + (-f'(z_0)z_0 + f(z_0))$$

so that we may approximate $f(z)$ by the linear function. Also,

$$|f(z) - f(z_0)| \approx |f'(z_0)|\,|z - z_0|\,. \tag{11.3}$$

In view of (11.3), "small" neighborhoods of z_0 are mapped onto roughly the same configuration, magnified by the factor $|f'(z_0)|$, see Figure 11.4. Hence, $f'(z_0)$ plays two roles in determining the geometric character of the image. According to (11.2), $\arg f'(z_0)$ measures the rotation; according to (11.3), $|f'(z_0)|$ measures (for points nearby) the magnification or distortion of the image.

An interesting comparison can now be made between the derivatives of real and complex functions. For real differentiable functions, the nonvanishing of the derivative is sufficient to guarantee that the function is one-to-one on an interval. This is not the case for complex functions on a domain. Even though

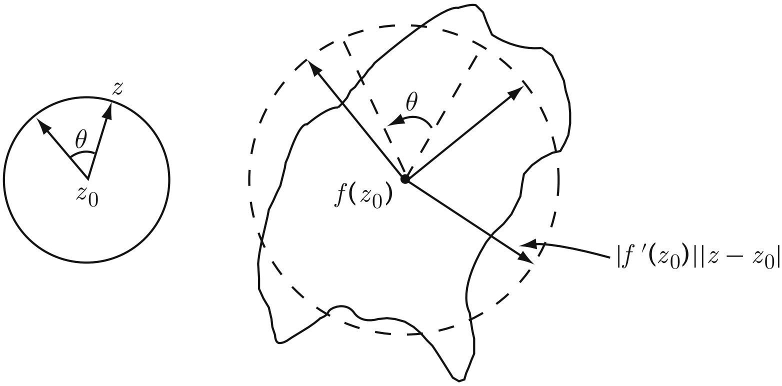

Figure 11.4.

the derivative of the entire function e^z never vanishes, we have $e^z = e^{z+2\pi i}$ for all z. Similarly, the entire function $f(z) = z^2$ is conformal on $\mathbb{C}\backslash\{0\}$. However, it is geometrically intuitive (Figure 11.4) that the nonvanishing of a derivative implies, at least locally, that the function is one-to-one. We now show this formally in the following form which gives a sufficient condition for the existence of a local inverse.

Theorem 11.2. *If $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$, then $f(z)$ is one-to-one in some neighborhood of z_0.*

Proof. Since $f'(z_0) \neq 0$ and $f'(z)$ is continuous at z_0, there exists a $\delta > 0$ such that

$$|f'(z) - f'(z_0)| < \frac{|f'(z_0)|}{2} \quad \text{for all } |z| < \delta.$$

Let z_1 and z_2 be two distinct points in $|z| < \delta$, and γ be a line segment connecting z_1 and z_2. Set $\phi(z) = f(z) - f'(z_0)z$ so that $|\phi'(z)| < |f'(z_0)|/2$ for all $|z| < \delta$. Now we have

$$|\phi(z_2) - \phi(z_1)| = \left| \int_\gamma \phi'(z)\, dz \right| < (|f'(z_0)|/2)|z_2 - z_1|,$$

or equivalently,

$$|f(z_2) - f(z_1) - f'(z_0)(z_2 - z_1)| < (|f'(z_0)|/2)|z_2 - z_1|.$$

Thus, by the triangle inequality, we obtain

$$|f(z_2) - f(z_1)| > (|f'(z_0)|/2)|z_2 - z_1| > 0.$$

That is, $f(z)$ is one-to-one in $|z| < \delta$. ■

The vanishing of a derivative does not preclude the possibility of a real function being one-to-one. Although the derivative of $f(x) = x^3$ is zero at the origin, the function is still one-to-one on the real line. That this cannot occur for complex functions is seen by

Theorem 11.3. *If $f(z)$ is analytic and one-to-one in a domain D, then $f'(z) \neq 0$ in D, so that f is conformal on D.*

Proof. If $f'(z) = 0$ at some point z_0 in D, then

$$f(z) - f(z_0) = \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$$

has a zero of order k ($k \geq 2$) at z_0. Since zeros of an analytic function are isolated, there exists an $r > 0$ so small that both $f(z) - f(z_0)$ and $f'(z)$ have no zeros in the punctured disk $0 < |z - z_0| \leq r$. Let $g(z) := f(z) - f(z_0)$, $C = \{z : |z - z_0| = r\}$ and

$$m = \min_{z \in C} |g(z)|.$$

Then, g has a zero of order k ($k \geq 2$) and $m > 0$. Let $b \in \mathbb{C}$ be such that $0 < |b - f(z_0)| < m$. Then, as $m \leq |g(z)|$ on C,

$$|f(z_0) - b| < |g(z)| \quad \text{on } C.$$

It follows from Rouche's theorem that $g(z)$ and

$$g(z) + (f(z_0) - b) = f(z) - b$$

have the same number of zeros inside C. Thus, $f(z) - b$ has at least two zeros inside C. Observe that none of these zeros can be at z_0. Since $f'(z) \neq 0$ in the punctured disk $0 < |z - z_0| \leq r$, these zeros must be simple and so, distinct. Thus, $f(z) = b$ at two or more points inside C. This contradicts the fact that f is one-to-one on D. ∎

We sum up our results for differentiable functions. In the real case, the nonvanishing of a derivative on an interval is a sufficient but not a necessary condition for the function to be one-to-one on the interval; whereas in the complex case, the nonvanishing of a derivative on a domain is a necessary but not a sufficient condition for the function to be one-to-one on the domain.

An analytic function $f : D \to \mathbb{C}$ is called *locally bianalytic* at $z_0 \in D$ if there exists a neighborhood N of z_0 such that restriction of f from N onto $f(N)$ is bianalytic. Clearly, a locally bianalytic map on D need not be bianalytic on D, as the example $f(z) = z^n$ ($n > 2$) on $\mathbb{C}\backslash\{0\}$ illustrates.

Combining Theorem 11.2 and Theorem 11.3 leads to the following criterion for local bianalytic maps.

Theorem 11.4. *Let $f(z)$ be analytic in a domain D and $z_0 \in D$. Then f is bianalytic at z_0 iff $f'(z_0) \neq 0$.*

A sufficient condition for an analytic function to be one-to-one in a simply connected domain is that it be one-to-one on its boundary. More formally, we have

Theorem 11.5. *Let $f(z)$ be analytic in a simply connected domain D and on its boundary, the simple closed contour C. If $f(z)$ is one-to-one on C, then $f(z)$ is one-to-one in D.*

Proof. Choose a point z_0 in D such that $w_0 = f(z_0) \neq f(z)$ for z on C. According to the argument principle, the number of zeros of $f(z) - f(z_0)$ in D is given by $(1/2\pi)\triangle_C \{f(z) - f(z_0)\}$. By hypothesis, the image of C must be a simple closed contour, which we shall denote by C' (see Figure 11.5). Thus the net change in the argument of $w - w_0 = f(z) - f(z_0)$ as $w = f(z)$ traverses the contour C' is either $+2\pi$ or -2π, according to whether the contour is traversed counterclockwise or clockwise. Since $f(z)$ assumes the value w_0 at least once in D, we must have

$$\frac{1}{2\pi}\triangle_C \{f(z) - f(z_0)\} = \frac{1}{2\pi}\triangle_C \{w - w_0\} = 1.$$

That is, $f(z)$ assumes the value $f(z_0)$ exactly once in D.

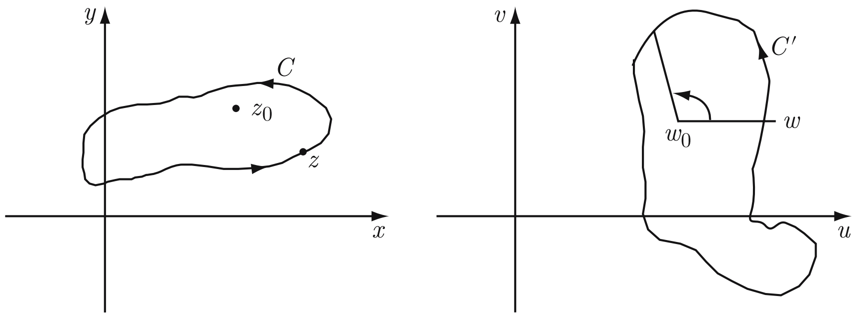

Figure 11.5.

This proves the theorem for all points z_0 in D at which $f(z) \neq f(z_0)$ when z is on C. If $f(z) = f(z_0)$ at some point on C, then the expression $\triangle_C \{f(z) - f(z_0)\}$ is not defined. We leave for the reader the completion of the proof in this special case. ■

In the proof of Theorem 11.1, we relied on the nonvanishing of the derivative. In Theorem 11.2, we see that every analytic function is locally one-to-one at points where the derivative is nonvanishing. More generally, it can be shown that if f is analytic at z_0 and f' has a zero of order k at z_0, then f is locally $(k+1)$-to-one. For example, if $f(z) = z^2$, then $f'(z)$ has a zero of order 1 at the origin and hence, it is two-to-one in any neighborhood of the origin.

We now examine the behavior of an analytic function in a neighborhood of a critical point, a point where the derivative vanishes. First we note that the angle of intersection of two smooth curves at a critical point of an analytic function is not the same as the angle of intersection of their images under f. If $f(z)$ is analytic and $f'(z)$ has a zero of order $k-1$ at $z=z_0$, then $f^{(j)}(z)=0$ for $j=1,\cdots,k-1$ and so we may write

$$f(z)=f(z_0)+a_k(z-z_0)^k+a_{k+1}(z-z_0)^{k+1}+\cdots.$$

Thus, $f(z)-f(z_0)=(z-z_0)^k g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0)=a_k\neq 0$. Consequently,

$$\arg[f(z)-f(z_0)]=k\arg(z-z_0)+\arg g(z). \tag{11.4}$$

Suppose θ is the angle that the tangent to a smooth curve C at z_0 makes with the x axis, and ϕ is the angle that the tangent to the image C' of the curve C at $f(z_0)$ makes with the u axis. If z approaches z_0 along the curve C, then $w=f(z)$ approaches $w_0=f(z_0)$ along the curve C', and so (11.4) yields

$$\phi=k\theta+\arg g(z_0). \tag{11.5}$$

Observe that (11.5) reduces to (11.2) in the special case when $k=1$. In general, the tangent to an image curve depends on the tangent to the original curve as well as on the order and argument of the first nonzero derivative at the point in question. Just as (11.2) led to Theorem 11.1, so (11.5) leads to

Theorem 11.6. *Suppose $f(z)$ is analytic at z_0, and that $f'(z)$ has a zero of order $k-1$ at z_0. If two smooth curves in the domain of f intersect at an angle θ, then their images intersect at an angle $k\theta$.*

Proof. Suppose that the tangents to the two curves make angles θ_1 and θ_2 with respect to the real axis. Then $\theta=\theta_2-\theta_1$ is the angle between the two curves. According to (11.5), the angle ϕ between their images is given by

$$\phi=k\theta_2+\arg g(z_0)-(k\theta_1+\arg g(z_0))=k\theta,\quad g(z_0)=\frac{f^{(k)}(z_0)}{k!}. \qquad \blacksquare$$

Combining Theorems 11.1 and 11.6, we see that an analytic function is conformal at a point if and only if it has a nonzero derivative at the point. Thus, an analytic function f is conformal on a domain D iff $f'(z)\neq 0$ on D.

It now pays to reexamine bilinear transformations, studied in Chapter 3, from a conformal mapping point of view. Recall that the transformation

$$w=f(z)=\frac{az+b}{cz+d}\quad (ad-bc\neq 0) \tag{11.6}$$

represents a one-to-one continuous mapping from the extended plane onto itself, with $f(-d/c)=\infty$ and $f(\infty)=a/c$. Since $f'(z)\neq 0$ $(ad-bc\neq 0)$, the mapping is conformal for all finite z, $z\neq -d/c$.

As we have seen, a circle or a straight line is mapped onto either a circle or a straight line, depending on which point is mapped onto the point at ∞. For instance, the inversion transformation $w = 1/z$ maps straight lines not passing through the origin onto circles. In particular, the lines $y = x + 1$ and $y = -x + 1$ are mapped, respectively, onto the circles

$$\left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 \quad \text{and} \quad \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2.$$

At first glance, Figure 11.6 is somewhat misleading. It shows a pair of straight lines that intersect at one point being mapped onto a pair of circles that intersect at two points. It should not be forgotten, however, that these straight lines also intersect at ∞. For both lines, the point $(0, 1)$ is mapped onto the point $(0, -1)$ while the point at ∞ is mapped onto the origin. The two lines intersect at right angles at $(0, 1)$ as do the two circles at $(0, -1)$. This is in harmony with Theorem 11.1.

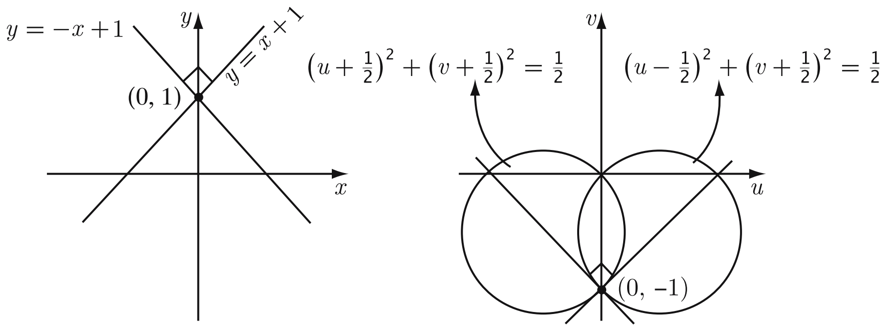

Figure 11.6.

But at what angle do the two lines intersect at ∞? We need the following definition: Two smooth curves in the extended plane are said to intersect at an angle α at ∞ if their images under the transformation $w = 1/z$ intersect at an angle α at the origin. Since the two circles in Figure 11.6 intersect at right angles at the origin, the lines $y = x+1$ and $y = -x+1$ intersect at right angles at ∞.

With this definition, we can show that all transformations of the form (11.6) are conformal at ∞. There are two cases to consider.

Case 1: Let $c \neq 0$. The behavior of f at ∞ is determined from the behavior of $f(1/z)$ at 0 in (11.6). Thus we consider

$$g(z) = f\left(\frac{1}{z}\right) = \frac{a/z + b}{c/z + d} = \frac{bz + a}{dz + c}.$$

Since $g'(0) = (bc - ad)/c^2 \neq 0$, it follows that $g(z)$ is conformal at $\zeta = 0$. But this means that $f(z)$ is conformal at $z = \infty$.

Case 2: Let $c = 0$. Then (11.6) is linear, and maps $z = \infty$ onto $w = \infty$. So we need to consider the expression $h(z) = 1/f(1/z)$ in (11.6):

$$w = h(z) = \frac{dz}{bz + a}.$$

Since $h'(0) = d/a \neq 0$, $h(z)$ is conformal at $z = 0$; that is, $f(z)$ is conformal at $z = \infty$. Hence, *a bilinear transformation is a one-to-one conformal mapping of the extended plane onto itself.*

Recall from Chapter 4 that the exponential function e^z maps lines parallel to the y axis onto circles centered at the origin and lines parallel to the x axis onto rays emanating from the origin. From elementary geometry we know that these two image curves must intersect at right angles (see Figure 11.7).

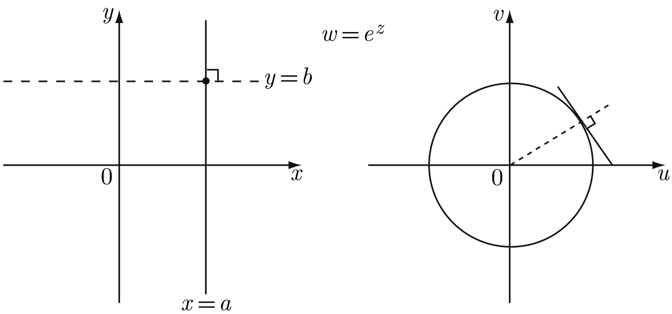

Figure 11.7.

Finally, consider the function $w = \cos z$, which maps lines parallel to the y axis onto ellipses and lines parallel to the x axis onto hyperbolas. According to Theorem 11.1, these conic sections must intersect at right angles (see Figure 11.8).

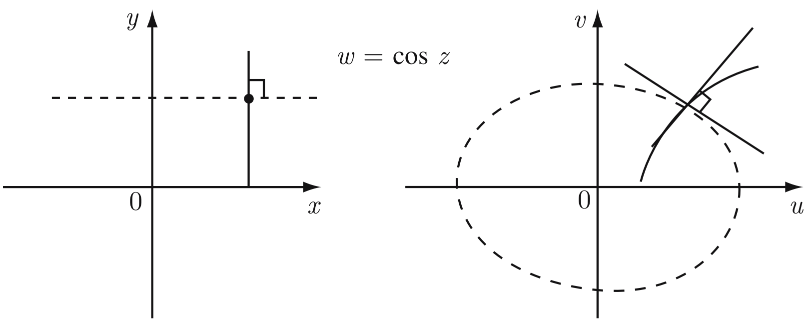

Figure 11.8.

Questions 11.7.

1. What is meant by a tangent to a point on a straight line?
2. Was it necessary to require the curves in Theorem 11.1 to be smooth?
3. Can nonanalytic functions be conformal?
4. What kind of functions are isogonal?
5. Why does the derivative play such a central role?
6. If a function is one-to-one in some neighborhood of each point in a domain, why does this not mean that the function is one-to-one in the domain?
7. If f is conformal on a domain D, is f always one-to-one on D?
8. If f is conformal on a domain D which is symmetric with respect to the real axis, is $\overline{f(\overline{z})}$ conformal on D?
9. What is the relationship between conformal and one-to-one?
10. At what angle do parallel lines intersect at ∞?
11. How might we define a function to be analytic at ∞?
12. Is the sum of conformal maps conformal? The product? The composition?

Exercises 11.8.

1. Given a complex number z_0 and an $\epsilon > 0$, show that there exists a function $f(z)$ analytic at z_0 with $f'(z_0) \neq 0$ and such that $f(z)$ is not one-to-one for $|z - z_0| < \epsilon$. Does this contradict Theorem 11.2?
2. Show that z^2 is one-to-one in a domain D if and only if D is contained in a half-plane whose boundary passes through the origin.
3. Find points at which the mapping defined by $f(z) = nz + z^n$ $(n \in \mathbb{N})$ is not conformal.
4. Prove that two smooth curves intersect at an angle α at ∞ if and only if their images under stereographic projection (see Section 2.4) intersect at an angle α at the north pole.
5. Show that $f(\overline{z})$ and $\overline{f(z)}$ are both isogonal at points where $f(z)$ is analytic with nonzero derivative.
6. If two straight lines are mapped by a bilinear transformation onto circles tangent to each other, show that the two lines must be parallel. Is the converse true?
7. Find the radius of the largest disk centered at the origin in which $w = e^z$ is one-to-one. Is the radius different if the disk is centered at an arbitrary point z_0?
8. For $f(z) = e^z$, find $\arg f'(z)$. Use this to verify that lines parallel to the y axis and x axis map, respectively, onto circles and rays.
9. Suppose $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$. Prove that a "small" rectangle containing z_0 and having area A is mapped onto a figure whose area is approximately $|f'(z_0)|^2 A$.
10. Either directly or by making use of Theorem 11.5, show that the function $w = z^n$ maps the ray $\arg z = \theta$ $(0 \leq \theta < 2\pi/n)$ onto the ray $\arg z = n\theta$.

11. If $f(z)$ is nonconstant and analytic in a domain D, show that $f'(z) = 0$ for only a countable number of points in D. Thus conclude that $f(z)$ is locally one-to-one and conformal at all but a countable number of points in D.
12. Show that $f(z) = z + 1/z$ is conformal except at $z = \pm 1$. With this in mind, review its mapping properties from Chapter 3.

11.2 Normal Families

We have previously seen significant differences between pointwise and uniform continuity as well as between pointwise and uniform convergence. Once again we encounter the contrast between local and global properties. This time, we shall require a uniformity to hold over a set consisting of a family of functions.

A family $\mathcal{F}$ of functions is said to be *uniformly bounded* on a set A if there exists a real number M such that $|f(z)| \leq M$ for all $f \in \mathcal{F}$ and all $z \in A$. Certainly the uniform boundedness of a family implies that each member of the family is bounded. On the other hand, each member of the sequence $\{f_n(z)\}$ of functions $f_n(z) = nz$ is bounded in the disk $|z| \leq R$, but there is no bound that works for every member of the family.

A family $\mathcal{F}$ of functions is said to be *locally uniformly bounded* on a set A if to each $z \in A$ there corresponds a neighborhood in which $\mathcal{F}$ is uniformly bounded. The sequence $f_n(z) = 1/(1 - z^n)$ is locally uniformly bounded, but not uniformly bounded in the disk $|z| < 1$. We have the following characterization:

Theorem 11.9. *A family $\mathcal{F}$ of functions is locally uniformly bounded in a domain D if and only if $\mathcal{F}$ is uniformly bounded on each compact subset of D.*

Proof. Let $\mathcal{F}$ be locally uniformly bounded and suppose K is a compact subset of D. For each point in K, choose a neighborhood in which $\mathcal{F}$ is uniformly bounded. This provides an open cover for K. According to the Heine–Borel theorem, there exists a finite subcover of K. That is, there are finitely many $z_i \in K$ and $\epsilon_i > 0$ such that $K \subset \bigcup_{i=1}^{n} N(z_i; \epsilon_i)$, where $|f(z)| \leq M_i$ for all $f \in \mathcal{F}$ and all $z \in N(z_i; \epsilon_i)$. Then $\mathcal{F}$ is uniformly bounded on K, having for a bound $M = \max\{M_1, M_2, \ldots, M_n\}$.

The converse is immediate from the fact that the closure of a neighborhood of a point is a compact set. ∎

By restricting ourselves to locally uniformly bounded families of analytic functions, we can obtain additional information.

Theorem 11.10. *Suppose $\mathcal{F}$ is a family of locally uniformly bounded analytic functions in a domain D. Then the family $\mathcal{F}^{(n)}$, consisting of the nth derivatives of all functions in $\mathcal{F}$, is also locally uniformly bounded in D.*

Proof. It suffices to prove this when $n = 1$, since then the result may be reapplied successively to each new class. Suppose for some z_0 in D that $|f(z)| \le M$ for each $f \in \mathcal{F}$ and all z inside or on the circle $C : |z - z_0| = r$ contained in D. Then for z in the smaller disk $|z - z_0| \le r/2$, Cauchy's integral formula yields

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

and so, as $|\zeta - z| \ge |\zeta - z_0| - |z - z_0| \ge r - r/2 = r/2$,

$$|f'(z)| \le \frac{1}{2\pi(r/2)^2} \int_C |f(\zeta)|\, |d\zeta| \le \frac{4M}{r}.$$

This shows that the family $\mathcal{F}'$ is locally uniformly bounded at z_0. Since z_0 was arbitrary, the proof is complete. ∎

We next extend the concept of uniform continuity. A family $\mathcal{F}$ of functions is said to be *equicontinuous* in a region R if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z_1) - f(z_0)| < \epsilon$ for all $f \in \mathcal{F}$ and all points $z_0, z_1 \in R$ satisfying $|z_1 - z_0| < \delta$. Observe that each member of an equicontinuous family is uniformly continuous. That is, for an equicontinuous family we can find a $\delta = \delta(\epsilon)$ that works for all points in the set as well as for all functions in the family.

It is possible for each member of a family to be uniformly continuous without the family being equicontinuous. To see this, set $f_n(z) = nz$. Each f_n is uniformly continuous on $|z| \le R$ because

$$|f_n(z_1) - f_n(z_0)| = n|z_1 - z_0| < \epsilon$$

whenever $|z_1 - z_0| < \epsilon/n = \delta$. But a δ cannot be chosen that works for all n. Hence the sequence $\{nz\}$ is not equicontinuous on $|z| \le R$.

There is an important relationship between locally uniformly bounded and equicontinuous families of analytic functions.

Theorem 11.11. *If $\mathcal{F}$ is a locally uniformly bounded family of analytic functions in a domain D, then $\mathcal{F}$ is equicontinuous on compact subsets of D.*

Proof. We prove the theorem in the special case that K is a closed disk contained in D. The proof for general compact subsets of D is similar to the proof of Theorem 11.9, and is left for the reader. By Theorem 11.10, the family $\mathcal{F}'$, consisting of the derivatives of functions in $\mathcal{F}$, is also locally uniformly bounded. In view of Theorem 11.9. we may therefore assume that $|f'(z)| \le M$ for all $f \in \mathcal{F}$ and all $z \in K$. Then for $z_0, z_1 \in K$, we have

$$|f(z_1) - f(z_0)| = \left| \int_{z_0}^{z_1} f'(z)\, dz \right| \le M|z_1 - z_0|,$$

where the path from z_0 to z_1 is taken to be the straight line segment. By choosing $\delta = \epsilon/M$ (ϵ arbitrary), we see that the family $\mathcal{F}$ is equicontinuous on the disk K. ∎

Remark 11.12. The converse of Theorem 11.11 is not true. The sequence $f_n(z) = z + n$ is equicontinuous on all compact subsets of the plane. In fact $f_n(z_1) - f_n(z_0) = z_1 - z_0$ for each n, so that $\delta = \epsilon$ may be chosen. However, $\{f_n(z)\}$ is not uniformly bounded in any neighborhood in $\mathbb{C}$. ●

In Chapter 2, we showed that every bounded sequence of complex numbers contains a convergent subsequence. Our goal in this section is to obtain analogous results for sequences of functions. It is not clear, at this point, what form of convergence is most reasonable or most applicable. To help clarify the situation, we need the following definition. A family $\mathcal{F}$ of functions is said to be *normal* in a domain D if every sequence $\{f_n\}$ in $\mathcal{F}$ contains a subsequence $\{f_{n_k}\}$ that converges uniformly on each compact subset of D.

As an example, the family consisting of the sequence $\{z^n\}$ is normal in the domain $|z| < 1$. In fact, the sequence itself converges uniformly to zero on every compact subset of $|z| < 1$. Note, however, that neither the sequence nor any subsequence converges uniformly in the whole domain.

Just as a bounded sequence may contain different subsequences that converge to different limits, so may a normal family contain different sequences that converge uniformly on compact subsets to different functions. To illustrate, set

$$f_n(z) = \begin{cases} z^n & \text{if } n \text{ odd}, \\ 1 - z^n & \text{if } n \text{ even}. \end{cases}$$

Then $\{f_{2n+1}\}$ converges uniformly to 0 and $\{f_{2n}\}$ converges uniformly to 1 on all compact subsets of $|z| < 1$.

A set of points E is said to be *dense* in a set A if every neighborhood of each point in A contains points of E. Every domain in the plane contains a dense sequence of points (for example, the set of points in the domain having both coordinates rational is countable, and so may be expressed as a sequence). Before proving the major result of this section, we need the following:

Lemma 11.13. *Suppose $\{f_n(z)\}$ is a sequence of analytic functions that is locally uniformly bounded in a domain D. If $\{f_n(z)\}$ converges at all points of a dense subset of D, then it converges uniformly on each compact subset of D.*

Proof. Given a compact set K contained in D, we wish to show that the sequence $\{f_n(z)\}$ converges uniformly on K. By Theorem 11.11, $\{f_n(z)\}$ is equicontinuous on K. Thus to each $\epsilon > 0$, there corresponds a $\delta > 0$ such that

$$|f_n(z) - f_n(z')| < \epsilon/3 \quad \text{for } |z - z'| < \delta, \tag{11.7}$$

where z, z' are any points in K and n is arbitrary. Since K is compact, finitely many, say p, neighborhoods of radius $\delta/2$ cover K. In each of these p neighborhoods, choose a point z_k $(k = 1, 2, \ldots, p)$ from the dense subset of K, at which $\{f_n\}$ converges. Next choose n and m large enough so that

$$|f_n(z_k) - f_m(z_k)| < \epsilon/3 \quad \text{for } k = 1, 2, \ldots, p. \tag{11.8}$$

In view of (11.7) and (11.8), we see that, to each z in K, there corresponds a z_k in K such that

$$\begin{aligned} |f_n(z) - f_m(z)| &\leq |f_n(z) - f_n(z_k)| + |f_n(z_k) - f_m(z_k)| + |f_m(z_k) - f_m(z)| \\ &< \epsilon. \end{aligned}$$

Hence the sequence $\{f_n(z)\}$ is uniformly Cauchy on K, and must therefore converge uniformly on K. ■

Note the lemma concludes that $\{f_n(z)\}$ is a normal family in D. We will now show, by a diagonalization process, that this conclusion is true without the assumption that the sequence converges on a dense subset.

Theorem 11.14. (Montel's Theorem) *If $\mathcal{F}$ is a locally uniformly bounded family of analytic functions in a domain D, then $\mathcal{F}$ is a normal family in D.*

Proof. Given a sequence $\{f_n\}$ of functions in $\mathcal{F}$, we must show that some subsequence of $\{f_n\}$ converges uniformly on compact subsets. Choose any sequence of points $\{z_k\}$ that is dense in D. According to Lemma 11.13, it suffices to construct a subsequence of $\{f_n\}$ that converges at each point of the sequence $\{z_k\}$. By hypothesis, the sequence $\{f_n(z_1)\}$ of complex numbers is bounded. Hence by the Bolzano–Weierstrass property (see Theorem 2.17), there exists a subsequence of $\{f_n\}$, which we shall denote by $\{f_{n,1}\}$, that converges at z_1. But the sequence of $\{f_{n,1}(z_2)\}$ of points is also bounded. Thus there is a subsequence $\{f_{n,2}\}$ of $\{f_{n,1}\}$ that converges at z_2. Since $\{f_{n,2}\}$ is a subsequence of $\{f_{n,1}\}$, it must also converge at z_1.

Continuing the process, for each positive integer m, we obtain the mth subsequence $\{f_{n,m}\}$ of $\{f_n\}$ so that it converges at $z_1, z_2, \ldots, z_m$. As seen in the chart below,

$$\begin{array}{cccccc} f_{1,1}(z), & f_{2,1}(z), & f_{3,1}(z), & \ldots & f_{m,1}(z), & \ldots \\ f_{1,2}(z), & f_{2,2}(z), & f_{3,2}(z), & \ldots & f_{m,2}(z), & \ldots \\ f_{1,3}(z), & f_{2,3}(z), & f_{3,3}(z), & \ldots & f_{m,3}(z), & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ f_{1,m}(z), & f_{2,m}(z), & f_{3,m}(z), & \ldots & f_{m,m}(z), & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

the mth sequence of complex functions converges at z_m and all preceding points of the sequence $\{z_k\}$. Now consider the sequence $\{f_{n,n}(z)\}$, which makes up the diagonal of the chart. For each fixed m, the sequence $\{f_{n,n}(z_m)\}$, $n \geq m$, is a subsequence of the convergent sequence $\{f_{n,m}(z_m)\}$, and hence converges. Therefore, $\{f_{n,n}(z)\}$ is a subsequence of $\{f_n\}$ that converges at all points of the sequence $\{z_k\}$. This completes the proof. ■

The Bolzano–Weierstrass theorem guarantees the existence of a limit point for every bounded infinite set of points, and consequently the existence of a convergent subsequence for every bounded sequence. Montel's theorem can be viewed as an "analytic function" analog to Bolzano's theorem. It guarantees, in some sense, the existence of a convergent sequence of functions associated with every locally uniformly bounded family of analytic functions.

Carrying the analogy one step further, both theorems suffer from the same deficiency. The limit point of Bolzano's theorem need not be a member of the set, while the convergent function of Montel's need not be a member of the normal family. For example, the sequence $\{z^n\}$ is a normal family in $|z| < 1$ because it converges uniformly to 0 on all compact subsets of $|z| < 1$. However, 0 is not a member of the family $\{z^n\}$.

Recall that a bounded set that contains all its limit points is compact. This leads to the following definition. A normal family $\mathcal{F}$ of functions is said to be *compact* if the uniform limits of all sequences converging in $\mathcal{F}$ are themselves members of $\mathcal{F}$.

Example 11.15. The family $\mathcal{F}$ of functions of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$

that are analytic with positive real part in the disk $|z| < 1$ is a compact, normal family. By Theorem 10.42, all functions $f \in \mathcal{F}$ satisfy the inequality

$$|f(z)| \le \frac{1+|z|}{1-|z|} \quad (|z| = r < 1).$$

Hence $\mathcal{F}$ is locally uniformly bounded and, by Montel's theorem, is normal. To show compactness, suppose a sequence $\{f_n\}$ of functions in $\mathcal{F}$ converges uniformly to a function g. We wish to show that $g \in \mathcal{F}$. By Theorem 8.16, g is analytic in $|z| < 1$. Since $f_n(0) = 1$ for every n, $g(0) = 1$. Since $\operatorname{Re} f_n(z) > 0$ for every n, $\operatorname{Re} g(z) \ge 0$ for $|z| < 1$. But then by the open mapping theorem, we must have $\operatorname{Re} g(z) > 0$ for $|z| < 1$. Thus $g \in \mathcal{F}$, and $\mathcal{F}$ is compact. ●

Questions 11.16.

1. What kinds of families of functions are locally uniformly bounded but not uniformly bounded?
2. Is the family of polynomials locally uniformly bounded on some set?
3. If $\mathcal{F}$ is a uniformly bounded family of analytic functions, is $\mathcal{F}^{(n)}$ also uniformly bounded?
4. If a family of functions is uniformly bounded at each point in a domain, is the family locally uniformly bounded?
5. Where, in the proof of Theorem 11.7, did we use the fact that the set K was a disk?

6. What is an important distinction between a dense sequence and a dense set?
7. What kinds of normal families have more than one subsequential limit function?
8. Can a normal family have infinitely many subsequential limit functions?

Exercises 11.17.

1. Suppose that for each point in a domain D there corresponds a neighborhood in which a family $\mathcal{F}$ is equicontinuous. Show that $\mathcal{F}$ is equicontinuous on compact subsets of D. Is $\mathcal{F}$ equicontinuous in D?
2. Show that the sequence $\{nz\}$ is not equicontinuous in any region.
3. If $\mathcal{F}$ is locally uniformly bounded family of analytic functions in a domain D, show that $\mathcal{F}'$, the family of functions consisting of the derivatives of functions in $\mathcal{F}$, is equicontinuous on compact subsets of D.
4. Suppose $\mathcal{F}$ is a normal family of analytic functions in the disk $|z| < 1$. Let $\mathcal{G}$ be the family of functions of the form $g(z) = \int_0^z f(\zeta)\, d\zeta$, where $f \in \mathcal{F}$. Show that $\mathcal{G}$ is normal in $|z| < 1$.
5. Show that the sequence $\{f_n(z)\}$ defined by

$$f_n(z) = \begin{cases} z^n & \text{if } n \text{ odd} \\ 1 - z^n & \text{if } n \text{ even,} \end{cases}$$

   forms a normal family in the disk $|z| < 1$.
6. Show that the family of functions of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $|a_n| \leq n$, is a compact normal family of analytic functions in the disk $|z| < 1$.
7. Let $\mathcal{F}$ be the family consisting of all functions $f(z)$ that are analytic in a domain D with $|f(z)| \leq M$ in D. Show that $\mathcal{F}$ is a compact, normal family in D.

11.3 Riemann Mapping Theorem

We have already discussed a number of examples of analytic functions between various domains of the complex plane. In some cases, we have given complete characterizations for mappings between certain domains such as disks and half-planes. Also, we know from the open mapping theorem that nonconstant analytic functions map domains into domains. Now, suppose D_1 and D_2 are simply connected domains. Then there is almost always an analytic function mapping D_1 onto D_2. We first discuss a "typical" exception. Suppose D_1 is the whole plane and D_2 is the disk $|z| < 1$. There can be no function analytic in the plane (entire) that maps onto the (bounded) disk $|z| < 1$, for, according to Liouville's theorem, constant functions are the only entire functions whose images are contained in the disk. Our major theorem of this section says that a one-to-one analytic mapping exists between any two simply connected domains, neither of which is the whole plane. Before proving this remarkable (existence) result, we shall need some preliminaries concerning *univalent* (a fancy term for one-to-one) functions.

Theorem 11.18. *Suppose $\{f_n(z)\}$ is a sequence of analytic, univalent functions defined in a domain D and converging uniformly on each compact subset of D to a nonconstant function $f(z)$. Then $f(z)$ is analytic and univalent in D.*

Proof. The analyticity of f follows from Theorem 8.16. To prove the univalence of f, assume there are distinct points z_0, z_1 in D for which $f(z_0) = f(z_1) = a$. We can find $r > 0$ (e.g., $r < |z_0 - z_1|/2$) so small that the closed disks centered at z_0 and z_1 with radius r are mutually disjoint and are contained in D. Assume further that $f(z) \neq a$ on the circles $C_0 : |z - z_0| = r$ and $C_1 : |z - z_1| = r$. This is possible because f is nonconstant. Let

$$m = \min_{z \in C_0 \cup C_1} |f(z) - a|.$$

Now choose n sufficiently large so that $|f_n(z) - f(z)| < m$ on $C_0 \cup C_1$. So, on $C_0 \cup C_1$,

$$|f(z) - a)| > m > |f_n(z) - f(z)| \text{ for large } n.$$

By Rouche's theorem, the function

$$f_n(z) - a = (f_n(z) - f(z)) + (f(z) - a)$$

has at least one zero inside C_0 and at least one zero inside C_1. This contradicts the univalence of $f_n(z)$ in D. ∎

Note that it is possible for the uniform limit of a sequence of univalent functions to be constant. For example, the univalent sequence $f_n(z) = z/n$ converges uniformly to $f(z) = 0$ on any compact subset of $\mathbb{C}$. Thus the uniform limit of a sequence of univalent functions need not be univalent.

Theorem 11.19. *Suppose $f(z)$ is analytic and univalent in a domain D, and that $g(z)$ is analytic and univalent on the image of D under $f(z)$. Then the composition function $g(f(z))$ is analytic and univalent in D.*

Proof. The analyticity of $g(f(z))$ follows from Theorem 5.6. To show univalence, suppose

$$g(f(z_0)) = g(f(z_1)) \text{ for } z_0, z_1 \in D.$$

By the univalence of g, we have $f(z_0) = f(z_1)$. From the univalence of f, $z_0 = z_1$ and the theorem is proved. ∎

Theorem 11.20. *Suppose f, mapping a domain D_1 onto D_2, is analytic and univalent in D_1. Then the inverse function g, defined by $g(f(z)) = z$ for all $z \in D_1$, is an analytic and univalent mapping from D_2 onto D_1.*

Proof. The univalence of g is an immediate consequence of the univalence of f. To show analyticity, fix a point $w_0 \in D_2$. Then $w_0 = f(z_0)$ for a unique $z_0 \in D_1$. Setting $w = f(z)$, we have

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}. \tag{11.9}$$

Since f maps open sets onto open sets (Theorem 9.55), g is continuous in D_2. Thus $z \to z_0$ as $w \to w_0$. By Theorem 11.3, $f'(z_0) \neq 0$. Hence we may take limits in (11.9) to obtain $g'(w_0) = g'(f(z_0)) = 1/(f'(z_0))$. Therefore g is analytic in D_2, and the theorem is proved. ∎

If f and g are analytic and univalent in domains D_1 and D_2, respectively, and map onto the disk $|z| < 1$, then $g^{-1}(f(z))$ is an analytic and univalent mapping from D_1 onto D_2 (see Figure 11.9).

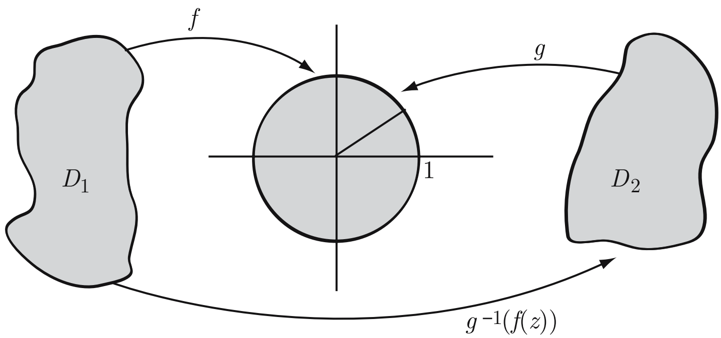

Figure 11.9.

Thus the set of domains that may be mapped analytically and univalently onto the interior of the unit disk can also be mapped analytically and univalently onto one another.

Suppose f is analytic and univalent in D and maps onto $|z| < 1$. Are there other functions with the same property? In general, there are infinitely many. To see this, recall from Section 3.2 (see Theorem 3.21) that all functions of the form

$$g(z) = e^{i\alpha} \frac{z - z_0}{1 - \overline{z}_0 z} \quad (|z_0| < 1, \ \alpha \ \text{ real}) \tag{11.10}$$

map the interior of the unit circle onto itself. Hence the functions $g(f(z))$ and $f(z)$ simultaneously map D onto $|z| < 1$. Our next result suggests conditions for establishing a unique mapping function.

Given a domain $D \subseteq \mathbb{C}$, we define the group of analytic automorphisms of D as follows: If $f : D \to D$ is an analytic function that is one-to-one and onto, then $f(z)$ is called an analytic/holomorphic automorphism of D. That is, $f(z)$

is called a conformal self-mapping of D. The set of all analytic automorphisms of D form what is called an "automorphism group" (with composition as the group operation) of D, and is denoted by $\mathrm{Aut}\,(D)$. The Schwarz lemma can be used to describe the automorphism groups of the upper half-plane, and the unit disk Δ (see also Theorems 3.18 and 3.21). It is easy to see that

$$\mathrm{Aut}_{\,a}(D) = \{f \in \mathrm{Aut}\,(D) : \ f(a) = a\}$$

forms a subgroup of the group $\mathrm{Aut}\,(D)$. Our next result is a reformulation of Theorem 3.21 in the language of automorphisms, but the new proof uses the Schwarz lemma.

Theorem 11.21. *We have*

$$\mathrm{Aut}\,(\Delta) = \left\{ e^{i\alpha}\left(\frac{z-a}{1-\overline{a}z}\right) : \ |a|<1, \ 0 \le \alpha \le 2\pi \right\}.$$

In particular, $\mathrm{Aut}_{\,0}(\Delta) := \{f \in \mathrm{Aut}\,(\Delta) : \ f(0) = 0\} = \{e^{i\alpha}z : \ \alpha \ \textit{real}\}$.

Proof. Let $a \in \Delta$, and

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}.$$

Obviously, φ_a is analytic for $|z| < 1/|a|$ ($|a| < 1$), $\varphi_a(\Delta) \subseteq \Delta$, and $\varphi_a(\partial\Delta) = \partial\Delta$. Moreover, φ_a is univalent on Δ and $(\varphi_a)^{-1} = \varphi_a$. Thus, $\varphi_a \in \mathrm{Aut}\,(\Delta)$. Also, the rotation $e^{i\theta}\varphi_a(z)$ ($\theta \in \mathbb{R}$) belongs to $\mathrm{Aut}\,(\Delta)$.

Conversely, let $f \in \mathrm{Aut}\,(\Delta)$. Then there exists a $b \in \Delta$ such that $f(0) = b$. Then $F(z)$ defined by $F = \varphi_b \circ f$ is also analytic and univalent in Δ, F maps Δ onto Δ, and $F(0) = 0$. By the Schwarz lemma,

$$|F(z)| \le |z| \ \text{ for } \ z \in \Delta.$$

Since F is analytic and one-to-one on Δ, F^{-1} exists on Δ. Moreover, F^{-1} is analytic and one-to-one on Δ with $F^{-1}(0) = 0$. We may again apply the Schwarz lemma to F^{-1} and obtain $|F^{-1}(w)| \le |w|$ for $w \in \Delta$. If we take $w = F(z)$, we get

$$|z| \le |F(z)| \ \text{ for } \ z \in \Delta.$$

Hence, $|F(z)| = |z|$, and so $F(z) = \lambda z$ with $|\lambda| = 1$, or

$$\varphi_b(f(z)) = \lambda z \ \text{or} \ \ f(z) = \varphi_b(\lambda z).$$

The desired result follows. ∎

Our next result suggests conditions for establishing a unique mapping function.

Lemma 11.22. *Suppose $f(z)$ is analytic and univalent in $|z| < 1$ and maps the disk onto itself. If $f(0) = 0$ and $f'(0) > 0$, then $f(z) = z$.*

Proof. As $\mathrm{Aut}_0(\Delta) := \{f \in \mathrm{Aut}(\Delta) : f(0) = 0\} = \{e^{i\alpha}z : \alpha \text{ real}\}$ and $f'(0) > 0$, the result follows. ■

Two domains D_1 and D_2 are said to be *conformally equivalent* if there is a bijective analytic function mapping D_1 onto D_2. Both the existence and method of finding it are two important components for conformal mappings. We start with a couple of examples illustrating conformal mappings between standard simply connected domains. It follows that conformally equivalent domains are homeomorphic but not the converse.

Example 11.23. We are interested in showing that the upper half disk $D = \{z : |z| < 1, \mathrm{Im}\, z > 0\}$ and the unit disk $\Delta = \{z : |z| < 1\}$ are conformally equivalent.

Step 1: We consider

$$w_1 = f_1(z) = \frac{1}{1-z}.$$

Then we know that f_1 transforms the unit disk Δ onto the right half-plane $\mathrm{Re}\, w_1 > 1/2$. Rewriting

$$w_1 = f_1(z) = \frac{1-\overline{z}}{|1-z|^2} = \frac{1-x+iy}{|1-z|^2},$$

we see that $\mathrm{Im}\, w_1 > 0$ iff $\mathrm{Im}\, z > 0$. Moreover, $z = 1$ is a pole of $f_1(z)$, the segment $[-1, 1]$ maps onto the half-line $[1/2, \infty)$ and the upper half circle $\{z : |z| = 1, \ \mathrm{Im}\, z > 0\}$ onto the half-line $\{w_1 : \mathrm{Re}\, w_1 = 1/2, \ \mathrm{Im}\, w_1 > 0\}$. Therefore, f_1 maps D onto $D_1 = \{w_1 : \mathrm{Re}\, w_1 > 1/2, \ \mathrm{Im}\, w_1 > 0\}$.

Step 2: The map $w_2 = f_2(w_1) = w_1 - 1/2$ maps the domain D_1 onto the first quadrant $D_2 = \{w_2 : \mathrm{Re}\, w_2 > 0, \ \mathrm{Im}\, w_2 > 0\}$.

Step 3: The map $w_3 = f_3(w_2) = w_2^2$ maps D_2 onto the upper half-plane $\mathbb{H}^+ = \{w_3 : \mathrm{Im}\, w_3 > 0\}$.

Step 4: The map $w = f_4(w_3) = \frac{w_3 - i}{w_3 + i}$ carries the upper half-plane $\mathbb{H}^+$ onto the unit disk $\{w : |w| < 1\}$. Finally a map f with the desired property is a composition

$$w = f(z) = (f_4 \circ f_3 \circ f_2 \circ f_1)(z) = f_4(f_3(f_2(f_1(z))))$$

which gives

$$w = f(z) = \frac{(1+z)^2 - 4i(1-z)^2}{(1+z)^2 + 4i(1-z)^2}.$$ ●

Example 11.24. Let $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$. Now we want to find a conformal map of D onto the unit disk Δ. As we can see from the picture, it suffices to focus on certain key points to understand the sequence of mappings considered here. If $w_1 = 1/(1-z)$, then $z = 1 - 1/w_1$ and

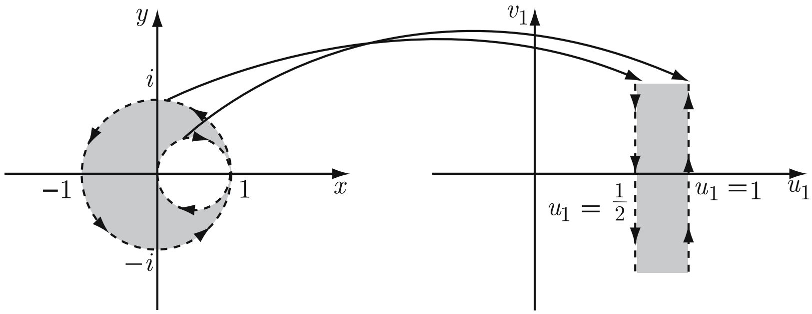

Figure 11.10. A conformal map of D onto the strip

$$\begin{cases} |z| < 1 & \Longleftrightarrow \operatorname{Re} w_1 > 1/2 \\ |z - 1/2| > 1/2 & \Longleftrightarrow \operatorname{Re} w_1 < 1. \end{cases}$$

Because of the basic property of Möbius transformations, it follows easily that f_1 maps D onto the strip $D_1 = \{w_1 : 1/2 < \operatorname{Re} w_1 < 1\}$. A similar explanation may be provided for other mappings. Finally, the composition

$$w = f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z)$$

gives the formula which does the required job, where

$$w_2 = f_2(w_1) = i\pi(w_1 - 1/2),\ w_3 = f_3(w_2) = e^{w_2},\ f_4(w_3) = \frac{w_3 - i}{w_3 + i}. \quad \bullet$$

We are now ready to formally state and prove the Riemann mapping theorem which is a classical example of existence theorems.

Theorem 11.25. (Riemann Mapping Theorem) *Suppose D is a simply connected domain, other than the whole plane, and z_0 is a point in D. Then there exists a unique function $f(z)$, analytic and univalent in D, which maps D onto the disk $|w| < 1$ in such a manner that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof. We first prove the uniqueness of the mapping function f. If g_1 and g_2 are two functions each of which maps D onto the unit disk $|w| < 1$ in the prescribed manner, then $h = g_2 \circ g_1^{-1}$ is an analytic and univalent mapping of the unit disk $|w| < 1$ onto itself. Furthermore,

$$h(0) = g_2(g_1^{-1}(0)) = g_2(z_0) = 0$$

and, because $g_1'(z_0) > 0$ and $g_2'(z_0) > 0$,

$$h'(0) = g_2'(g_1^{-1}(0))(g_1^{-1})'(0) = \frac{g_2'(z_0)}{g_1'(z_0)} > 0.$$

Hence, by Lemma 11.22, h is the identity function. That is, $g_1(z) = g_2(z)$ and uniqueness is proved.

To prove existence of the mapping function, we first show that there is an analytic and univalent function mapping D *into* the disk $|w| < 1$. Since D is not the whole plane $\mathbb{C}$, there is a point $a \in \mathbb{C}\backslash D$. If there is actually a disk $|z-a| < \epsilon$ outside of D, then $|z-a| > \epsilon$ for all points z in D. In this case,

$$w = \frac{\epsilon}{z-a}$$

is an analytic and univalent function that maps all points of D into the unit disk $|w| < 1$. Thus, the proof follows if D is a bounded domain. However, if D is unbounded, then it is possible that the complement of D does not contain any disk. For instance, D might be the plane minus a ray from some point z_0 to ∞. This kind of difficulty will be avoided by considering a branch of the square root function, which maps a domain onto one "half" its size.

According to Corollary 7.52, if $a \in \mathbb{C}\backslash D$, then there exists an analytic function $\phi : D \to \mathbb{C}$, called analytic branch of $(z-a)^{1/2}$ with $\phi^2(z) = z-a$ so that $\phi(z) = \sqrt{z-a}$. Furthermore, $\phi(z)$ is univalent in D. For if $\phi(z_1) = \phi(z_2)$ for $z_1, z_2 \in D$, then

$$[\phi(z_1)]^2 = [\phi(z_2)]^2, \quad \text{i.e.,} \quad z_1 - a = z_2 - a.$$

Now let $D' = \phi(D)$. Then D' is simply connected since D is simply connected. Then the complement of D' contains a disk. To see this, we will show that points b and $-b$ cannot simultaneously be in D'. For if they are, then there exist two points z_1 and z_2 in D such that $\phi(z_1) = b$ and $\phi(z_2) = -b$. Now,

$$\begin{aligned} \phi(z_1) = -\phi(z_2) &\Longrightarrow [\phi(z_1)]^2 = [\phi(z_2)]^2 \\ &\Longrightarrow z_1 - a = z_2 - a, \quad \text{i.e.,} \quad z_1 = z_2 \\ &\Longrightarrow b = -b, \quad \text{i.e.,} \quad \phi(z_1) = 0 = \phi(z_2) \\ &\Longrightarrow z_2 = a \in \mathbb{C}\backslash D, \end{aligned}$$

contradicting the fact that z_1 and z_2 are distinct.

Next choose a point $w_0 \in D'$ and an $\epsilon > 0$ so that the disk $|w - w_0| < \epsilon$ is contained in D'. Then the disk $|w + w_0| < \epsilon$ is contained in the complement $\mathbb{C}\backslash D'$. Hence the function

$$\psi(w) = \frac{\epsilon}{w + w_0}$$

maps D' into the unit disk, because $|w + w_0| > \epsilon$ for all $w \in D'$. Therefore, the composition

$$f(z) = \psi(\phi(z)) = \frac{\epsilon}{\phi(z) + w_0}$$

is analytic and univalent in D and maps D into the unit disk. By a suitable bilinear transformation (fill in details!), we can transform this function into a function $f_0(z)$ satisfying the additional conditions $f_0(z_0) = 0$ and $f_0'(z_0) > 0$.

Let $\mathcal{F}$ denote the family of all analytic functions $g: D \to \mathbb{C}$ such that $g(z)$ is univalent in D, $g(z_0) = 0$, $g'(z_0) > 0$, and satisfies $|g(z)| < 1$ for all z in D. The family $\mathcal{F}$ is nonempty because $f_0(z) \in \mathcal{F}$. Certainly the function whose existence we are determined to prove must also be in the family $\mathcal{F}$. It will be shown that the desired function has a larger derivative at z_0 than any other function in $\mathcal{F}$. To show the existence of a function in $\mathcal{F}$ with a maximum derivative at z_0, we will rely on the theory of normal families.

Since the family $\mathcal{F}$ is locally uniformly bounded (in fact, uniformly bounded) in D, it follows from Theorem 11.14 that $\mathcal{F}$ is a normal family. Set

$$A = \operatorname{lub}\{g'(z_0) : g \in \mathcal{F}\}.$$

Then, $A > 0$ because $g'(z_0) > 0$ for each $g \in \mathcal{F}$. But A may be infinite. By the definition of A, there is a sequence $\{f_n\}$ of functions in $\mathcal{F}$ such that $f_n'(z_0) \to A$. By the normality of $\mathcal{F}$, there exists a subsequence $\{f_{n_k}\}$ that converges uniformly on the compact subsets of D to an analytic function $f(z)$. An application of Corollary 8.18 shows that $f'(z_0) = A$, so that A is finite. Since $f'(z_0) \geq f_0'(z_0) > 0$, the function $f(z)$ is not constant in D. It thus follows from Theorem 11.18 that $f(z)$ is univalent and, consequently, a member of $\mathcal{F}$.

We shall now show that this f maps D onto the unit disk, and so it is the required function. For the sake of obtaining a contradiction we suppose that $f(D)$ is not the whole unit disk $|w| < 1$. Then $f(z) \neq \alpha$ for some α with $|\alpha| < 1$. By the definition of analytic branch of square roots, there exists an analytic function $F(z)$ in D so that

$$F(z)^2 = \frac{f(z) - \alpha}{1 - \overline{\alpha} f(z)}.$$

The univalence of $F(z)$ follows from the univalence of $f(z)$, and the inequality $|F(z)| < 1$ follows from the inequality $|f(z)| < 1$. However, $F(z)$ is not properly normalized. We therefore consider the function

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)} \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} F(z)},$$

which satisfies $G(z_0) = 0$ and $G'(z_0) > 0$, so that $G(z) \in \mathcal{F}$. Moreover,

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{1 + |\alpha|}{2\sqrt{|\alpha|}} A > A = f'(z_0),$$

contradicting the maximality of $f'(z_0)$. Thus $f(z)$ omits no values inside the unit disk, and the proof is complete. ■

Remark 11.26. Since univalence in a domain guarantees a nonvanishing derivative, the Riemann mapping theorem shows that any two simply connected domains (neither of which is the plane) are conformally equivalent.

In the proof of Theorem 11.25, we assumed that an analytic, univalent function maps simply connected domains onto simply connected domains. In elementary topology, it is proved that the one-to-one continuous image of a simply connected domain cannot be multiply connected. Thus, we conclude that no simply connected domain can be conformally equivalent to a multiply connected domain. ●

Remark 11.27. Recall that a bilinear transformation maps circles and straight lines onto circles and straight lines. Hence any conformal mapping of a domain, other than a disk or a half-plane, onto the interior of the unit circle must be accomplished by a function other than a bilinear transformation. Furthermore, by the uniqueness property of the Riemann mapping theorem, no univalent function other than a bilinear transformation can map a disk or a half-plane onto the interior of the unit circle.

At this point, we must reflect on a sobering thought. The Riemann mapping theorem, like many existence theorems, has the drawback of not furnishing much insight into the actual construction. Therefore, given two "unfamiliar" simply connected domains, we must plod along as before to develop techniques for determining an appropriate mapping function. ●

Remark 11.28. The mapping of the interior of an arbitrary polygon onto the interior of the unit circle, whose existence is guaranteed by the theorem, can be found explicitly. This is accomplished in several stages. The *Schwarz–Christoffel formula* gives an analytic and univalent mapping of the upper half-plane onto the interior of an arbitrary polygon. For a complete discussion of the Schwarz–Christoffel transformation, we refer the reader to Nehari [N]. Composing the inverse of such a mapping with a bilinear transformation from the upper half-plane onto the open unit disk (see Section 3.3) gives the desired mapping. ●

Example 11.29. Let $f : \Omega \to \Omega$ be analytic in a simply connected domain Ω ($\neq \mathbb{C}$) having a fixed point in Ω. Then it can easily be shown that $|f'(a)| \leq 1$, and if $|f'(a)| = 1$, then f is actually a homeomorphism from Ω onto Ω.

The Riemann mapping theorem assures the existence of a bijective conformal map $\phi : \Omega \to \Delta$ such that $\phi(a) = 0$. Then we see that g defined by

$$g(z) = \phi \circ f \circ \phi^{-1}(z)$$

maps Δ into Δ and satisfies the hypothesis of the Schwarz lemma. Now, we easily see that $g'(0) = f'(a)$ and so $|f'(a)| \leq 1$, because $|g'(0)| \leq 1$. Moreover,

$$\begin{aligned} |f'(a)| = 1 &\Longrightarrow |g'(0)| = 1 \\ &\Longrightarrow g(z) = e^{i\alpha} z \quad \text{(by the Schwarz lemma)} \\ &\Longrightarrow \phi \circ f \circ \phi^{-1}(z) = e^{i\alpha} z \\ &\Longrightarrow f(z) = \phi^{-1}(e^{i\alpha}\phi(z)) \end{aligned}$$

which implies that f must be a bijective mapping from Ω onto Ω, because $\phi : \Omega \to \Delta$ and $\phi^{-1} : \Delta \to \Omega$ are bijective maps. ●

Questions 11.30.

1. Must the convergence be uniform in Theorem 11.18 in order for the conclusion to be valid?
2. Are there conformal mappings from multiply connected domains onto multiply connected domains?
3. If $f(z)$ is analytic and conformal in a domain D_1 and maps D_1 onto D_2, are D_1 and D_2 conformally equivalent?
4. What other initial conditions could we have prescribed in the Riemann mapping theorem to guarantee uniqueness?
5. Does there exist a one-to-one conformal mapping from the unit disk onto the disk minus the origin?
6. If two domains are conformally equivalent, what can be said about their boundaries?
7. Does there always exist an analytic function which maps a simply connected domain $\Omega (\neq \mathbb{C})$ into the unit disk $|z| < 1$?
8. Let $\Omega\, (\neq \mathbb{C})$ be a simply connected domain and let $\mathcal{F}$ be the set of all one-to-one analytic functions which map Ω into the unit disk $|z| < 1$, and $a \in \Omega$. If $f \in \mathcal{F}$ and is not onto, is there a function $g \in \mathcal{F}$ such that $|g'(a)| > |f'(a)|$?
9. Are the plane $\mathbb{C}$ and the unit disk $|z| < 1$ conformally equivalent? Are they homeomorphic?
10. Are the plane $\mathbb{C}$ and the upper half-plane $\operatorname{Im} w > 0$ conformally equivalent? Are they homeomorphic?
11. In the statement of the Riemann mapping theorem, why do we require the domain D to be a proper subset of $\mathbb{C}$? Does the theorem still hold if we remove that assumption?
12. Does the proof of the Riemann mapping theorem use the fact that every nonvanishing analytic function in a simply connected domain D admits analytic square root function in D?
13. Where, in the proof of the Riemann mapping theorem, did we require the domain to be simply connected?
14. Why was it necessary to first show that some function mapped the domain *into* the unit disk?
15. Why does the function $G(z)$, constructed in the proof of the Riemann mapping theorem, work?
16. What is a conformal map between the upper half-plane $\mathbb{H}^+ = \{z : \operatorname{Im} z > 0\}$ and $\mathbb{C} \setminus [0, \infty)$?
17. What is a conformal map between the right half-plane $D_1 = \{z : \operatorname{Re} z > 0\}$ and $D_2 = \{z : |\operatorname{Arg} z| < \pi/8\}$?
18. What is a conformal map between the strip $D_1 = \{z : 0 < \operatorname{Im} z < \pi/2\}$ and the upper half-plane $\mathbb{H}^+$?

19. What is a conformal map between the strip $D_1 = \{z : 0 < \operatorname{Im} z < \alpha\}$ and the upper half-plane $\mathbb{H}^+$?
20. What is a conformal map between the infinite strip $|\operatorname{Re} z| < \pi/2$ and the unit disk $|w| < 1$?
21. What is a conformal map between the unit disk $|z| < 1$ and $\mathbb{C} \setminus \Delta$?

Exercises 11.31.

1. Suppose $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$. Show that there exist neighborhoods U and V of z_0 and $f(z_0)$, respectively, such that $f(z)$ is a univalent mapping from U onto V.
2. Show that the plane is not conformally equivalent to the upper half-plane. More generally, show that the plane is only conformally equivalent to itself.
3. Let $D_1 = \{z : 0 < \operatorname{Re} z,\ \operatorname{Im} z < \infty\}$ and $D_2 = \{w : \operatorname{Im} w > 0\}$ be the open first quadrant and the upper half-plane, respectively. By the Riemann mapping theorem D_1 and D_2 are conformally equivalent. Show that $f(z) = z^2$ does this job.
4. Let $D_1 = \{z : |\operatorname{Re} z| < \pi/2\}$ and $D_2 = \{w : \operatorname{Re} w > 0\}$. Show that $f : D_1 \to D_2$ given by $f(z) = e^{iz}$ is conformal.
5. Even though the interior of a square can be mapped conformally onto the interior of a circle, show that no square can be mapped conformally onto a circle.
6. Let D_1 be the annulus $0 < r_1 < |z| < R_1$ and D_2 be the annulus $0 < r_2 < |z| < R_2$. If
$$\frac{R_1}{r_1} = \frac{R_2}{r_2},$$
construct an analytic and univalent function that maps D_1 onto D_2.
7. Suppose D_1 and D_2 are conformally equivalent, and that D_2 and D_3 are conformally equivalent. Show that D_1 and D_3 are conformally equivalent.

11.4 The Class $\mathcal{S}$

We continue our investigation of univalent functions—a specialized topics in complex analysis. Analytically, a univalent function has a nonvanishing derivative (Theorem 11.3); geometrically, a univalent function maps simple curves onto simple curves.

Functions that are both analytic and univalent have a nice property of mapping simply connected domains onto simply connected domains. By the Riemann mapping theorem, we can associate a univalent function defined in an arbitrary simply connected domain (other than the whole plane) with one defined in the unit disk. Therefore, we shall restrict the domain on which these functions are defined to the disk $|z| < 1$. Our results will have a nicer form if we also assume that the function has a zero (hence its only zero) at the origin and that its derivative is equal to one at the origin. Since the derivative of a univalent function never vanishes, every univalent function $h(z)$ may be reduced to a function of this form by replacing it with

$$f(z) = \frac{h(z) - h(0)}{h'(0)}.$$

We shall denote by $\mathcal{S}$ the class of all functions $f(z)$ that are analytic and univalent in the unit disk $|z| < 1$, and are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus a function $f(z)$ in $\mathcal{S}$ has the power series representation

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (|z| < 1).$$

We shall denote by $\mathcal{T}$ the class of all functions of the form

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$$

that are analytic and univalent in the domain $|z| > 1$. The following relationship will enable us to deduce information about $\mathcal{S}$ from information about $\mathcal{T}$.

Theorem 11.32. *If* $f(z) \in \mathcal{S}$, *then* $1/f(1/z) \in \mathcal{T}$.

Proof. First suppose $1/f(1/z_1) = 1/f(1/z_2)$ ($|z_1| > 1$, $|z_2| > 1$). Then $f(1/z_1) = f(1/z_2)$, where $|1/z_1| < 1$ and $|1/z_2| < 1$. The univalence of $1/f(1/z)$ ($|z| > 1$) now follows from the univalence of $f(z)$ ($|z| < 1$). The analyticity of $1/f(1/z)$ will be a consequence of the analyticity of $f(z)$ if we can show that $f(1/z) \neq 0$ for $|z| > 1$. If $f(1/z_0) = 0$ for $0 < |1/z_0| < 1$, then $f(0) = f(1/z_0) = 0$, contradicting the univalence of $f(z)$ for $|z| < 1$. Hence $1/f(1/z) \in \mathcal{T}$, and the proof is complete. ■

The next theorem, because of its proof rather than its statement, is known as the *area theorem.*

Theorem 11.33. *If* $g(z) = z + b_0 + (b_1/z) + (b_2/z^2) + \cdots$ *is in* $\mathcal{T}$, *then* $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

Proof. The univalent function $g(z)$ maps the circle $|z| = r > 1$ onto a simple closed contour C. Set $g(z) = u(z) + iv(z)$. The area of the region R enclosed by C, denoted by $A(r)$, is

$$A(r) = \iint_R du\, dv.$$

Note that $A(r) > 0$ for each $r > 1$. If we now let $P(u, v) = -v/2$ and $Q(u, v) = u/2$, an application of Green's theorem yields

$$A(r) = \frac{1}{2}\int_C u\,dv - v\,du = \frac{1}{2}\int_0^{2\pi}\left(u\frac{\partial v}{\partial \theta} - v\frac{\partial u}{\partial \theta}\right)d\theta, \tag{11.11}$$

where $A(r) > 0$. By Exercise 5.2(13), we have $g'(z) = (1/iz)(\partial g/\partial \theta)$. To evaluate the line integral of (11.11), consider the integral

$$\begin{aligned}\frac{1}{2}\int_{|z|=r}\overline{g(z)}g'(z)\,dz &= \frac{1}{2}\int_0^{2\pi}(u-iv)\left[\frac{1}{iz}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\right] iz\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left(u\frac{\partial u}{\partial \theta} + v\frac{\partial v}{\partial \theta}\right)d\theta + \frac{i}{2}\int_0^{2\pi}\left(u\frac{\partial v}{\partial \theta} - v\frac{\partial u}{\partial \theta}\right)d\theta,\end{aligned} \tag{11.12}$$

whose imaginary part corresponds to $A(r)$. In order to simplify (11.12), we write

$$\int_{|z|=r}\overline{g(z)}g'(z)\,dz = \int_{|z|=r}\left(\overline{z} + \sum_{m=0}^{\infty}\overline{b}_m(\overline{z})^{-m}\right)\left(1 - \sum_{n=1}^{\infty}nb_nz^{-n-1}\right)dz,$$

and note that

$$\int_{|z|=r}(\overline{z})^{-m}z^{-n-1}\,dz = \begin{cases} 2\pi i r^{-2m} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

This leads to the identity

$$\begin{aligned}\frac{1}{2}\int_{|z|=r}\overline{g(z)}g'(z)\,dz &= \frac{1}{2}\int_{|z|=r}\overline{z}\,dz - \frac{1}{2}\int_{|z|=r}\frac{\sum_{n=1}^{\infty}n|b_n|^2r^{-2n}}{z}\,dz \\ &= \pi i\left(r^2 - \sum_{n=1}^{\infty}\frac{n|b_n|^2}{r^{2n}}\right).\end{aligned}$$

Therefore (11.12) is purely imaginary, and

$$A(r) = \frac{1}{2}\int_0^{2\pi}\left(u\frac{\partial v}{\partial \theta} - v\frac{\partial u}{\partial \theta}\right)d\theta = \pi\left(r^2 - \sum_{n=1}^{\infty}\frac{n|b_n|^2}{r^{2n}}\right). \tag{11.13}$$

Since $A(r) > 0$, we have

$$r^2 - \sum_{n=1}^{\infty}\frac{n|b_n|^2}{r^{2n}} > 0 \quad (r > 1). \tag{11.14}$$

But (11.14) is valid for every $r > 1$ so that the result follows upon letting $r \to 1^+$. ■

Remark 11.34. According to (11.13), the area enclosed by the image of the circle $|z| = r$ is at most πr^2 (the area enclosed by the circle), with equality only for $g(z) = z + b_0$. Furthermore, equality in the conclusion of the theorem holds if and only if the area enclosed by the image of $|z| = r > 1$ becomes arbitrarily small as $r \to 1$. ●

Remark 11.35. If $b_1 = 1$, then $b_n = 0$ for $n > 1$. Recall that the properties of $g(z) = z + 1/z$ were extensively studied in Section 3.3. In particular, this function was shown to map $|z| = r > 1$ onto an ellipse, and the ellipse approaches the linear segment $[-2, 2]$ as r approaches 1. ●

The coefficient bound for functions in $\mathcal{T}$, as expressed by the area theorem, will enable us to obtain a coefficient bound for functions in $\mathcal{S}$. But first we need the following:

Lemma 11.36. *If $f(z) \in \mathcal{S}$, then $z\sqrt{f(z^2)/z^2} \in \mathcal{S}$.*

Proof. Set $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$f(z^2) = z^2[1 + a_2 z^2 + a_3 z^4 + \cdots] := z^2 h(z),$$

where $h(z)$ is analytic and never vanishes in the unit disk. Therefore, choosing a branch of $(h(z))^{1/2}$ with $(h(0))^{1/2} = 1$, we see that $g(z)$ defined by

$$g(z) = z\sqrt{\frac{f(z^2)}{z^2}} = z\sqrt{1 + a_2 z^2 + a_3 z^4 + \cdots} \tag{11.15}$$

is analytic with $g(0) = 0$ and $g'(0) = 1$. To prove that $g(z)$ is univalent, suppose $g(z_1) = g(z_2)$. Then $f(z_1^2) = f({z_2}^2)$, and the univalence of $f(z)$ shows that $z_1^2 = z_2^2$, that is, $z_1 = \pm z_2$. But from (11.15), we see that $g(z)$ is an odd function. Hence, $z_1 = -z_2$ implies $g(z_1) = -g(z_2)$, which is a contradiction unless $z_1 = z_2 = 0$. Therefore $z_1 = z_2$, thus establishing the univalence of $g(z)$. ■

Remark 11.37. It was necessary to write $z\sqrt{f(z^2)/z^2}$ instead of $\sqrt{f(z^2)}$ because $f(z^2)$ has a zero at the origin, which makes the expression

$$\sqrt{f(z^2)} = e^{(1/2)\,\mathrm{Log}\, f(z^2)}$$

meaningless. ●

Theorem 11.38. *If $f(z) = z + a_2 z^2 + \cdots$ is in $\mathcal{S}$, then $|a_2| \leq 2$.*

Proof. By Lemma 11.36, $g(z) = z\sqrt{f(z^2)/z^2} \in \mathcal{S}$. We can verify from the expansion in (11.15) that $g'''(0) = 3a_2$. Thus we may write

$$g(z) = z + \frac{a_2}{2} z^3 + \cdots .$$

In view of Theorem 11.32, the Laurent expansion for $1/g(1/z)$ shows that

$$\frac{1}{g(1/z)} = \frac{1}{(1/z)[1 + (a_2/2)z^2 + \cdots]} = z - \frac{a_2}{2}\frac{1}{z} + \cdots \in \mathcal{T}.$$

Applying Theorem 11.33, we find that $|a_2/2|^2 \leq 1$, i.e., $|a_2| \leq 2$. ■

Remark 11.39. Retracing the steps in the proof, we can determine when equality holds. For if $a_2 = 2e^{i\alpha}$, α real, then $1/g(1/z) = z - e^{i\alpha}/z$. But this means that $g(z) = z/(1-e^{i\alpha}z)^2 = z\sqrt{f(z^2)/z^2}$, so that

$$f(z) = \frac{z}{(1-e^{i\alpha}z)^2} = z + 2e^{i\alpha}z^2 + 3e^{2i\alpha}z^3 + \cdots . \tag{11.16}$$

For each $\alpha \in \mathbb{R}$, this function is known as the Koebe function Moreover, it is easy to verify that the functions f maps $|z| < 1$ onto the w plane cut along the ray with constant argument from $-\frac{1}{4}e^{-i\alpha}$ to ∞. ●

The functions in (11.16) are extremal for Theorem 11.38 in the sense that there is equality on the bound for the second coefficient. Impressed by the fact that the Koebe function appears in many problems concerning the class $\mathcal{S}$, Bieberbach asked whether we always have $|a_n| \leq n$. This give rise to the famous

Bieberbach Conjecture. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{S}$, then $|a_n| \leq n$ for every n.*

Theorem 11.38 proves the conjecture for $n = 2$. Although stated in 1916, the conjecture was verified only for the values of n up to $n = 7$ until Louis de Branges proved the whole conjecture in 1985. For all n the maximization of $|a_n|$ is achieved only by the Koebe function. A large amount of research in the theory of univalent functions is centered on the Bieberbach conjecture.

The result for $n = 2$ can be used to prove the following elegant theorem which shows that this mapping property is, in a sense, extremal.

Theorem 11.40. *If $f(z) \in \mathcal{S}$ and $f(z) \neq c$ for $|z| < 1$, then $|c| \geq \frac{1}{4}$.*

Proof. Set $f(z) = z + a_2 z^2 + \cdots$. Since $f(z) \neq c$, the function

$$g(z) = \frac{cf(z)}{c - f(z)} = z + \left(a_2 + \frac{1}{c}\right) z^2 + \cdots$$

is also in $\mathcal{S}$. Applying Theorem 11.38 to $g(z)$, we get $|a_2 + (1/c)| \leq 2$. Thus, $|1/c| - |a_2| \leq |(1/c) + a_2| \leq 2$. Now, applying Theorem 11.38 to $f(z)$, we have $|1/c| \leq 2 + |a_2| \leq 4$, and the result follows. ■

Remark 11.41. Theorem 11.40 is known as a *covering theorem* or *Koebe one-quarter theorem*. It says that every function in $\mathcal{S}$ maps the unit disk $|z| < 1$ onto a domain in the w plane that contains the disk $|w| < \frac{1}{4}$. This result has a lot of interesting applications in many other parts of complex analysis. By the inverse function theorem (also by the open mapping theorem), $f(|z| < 1)$ contains an open neighborhood of the origin (since $f(0) = 0$ and $f'(0) \neq 0$). The Koebe $\frac{1}{4}$–theorem actually estimates the size of this neighborhood. ●

Finally, we end the section with the following results which provides a sufficient condition for an analytic function to be univalent.

Theorem 11.42. *If $f(z)$ is analytic in a convex domain D, and* $\operatorname{Re} f'(z) > 0$ *in D, then $f(z)$ is univalent in D.*

Proof. Choose distinct points $z_0, z_1 \in D$. Then the straight line segment $z = z_0 + t(z_1 - z_0)$, $0 \le t \le 1$, must lie in D. Integrating along this path, we get

$$f(z_1) - f(z_0) = \int_{z_0}^{z_1} f'(z)\,dz = \int_0^1 f'(z_0 + t(z_1 - z_0))\,(z_1 - z_0)\,dt.$$

Dividing by $z_1 - z_0$ and taking real parts, we have

$$\operatorname{Re}\left\{\frac{f(z_1) - f(z_0)}{z_1 - z_0}\right\} = \operatorname{Re}\left\{\int_0^1 f'(z_0 + t(z_1 - z_0))\,dt\right\} > 0.$$

Thus $f(z_1) \neq f(z_0)$, and $f(z)$ is univalent in D. ■

Questions 11.43.

1. What kind of results could have been obtained in this section if the functions had not been normalized?
2. What was the importance of the class $\mathcal{T}$?
3. Why was a bound on $|a_2|$ so useful?
4. Can $|a_2| = 2$ if $f(z)$ is a bounded function in $\mathcal{S}$?
5. Why is the Koebe function extremal for so many theorems?
6. For each n, are we guaranteed the existence of a function in $\mathcal{S}$ for which the absolute value of its nth coefficient is at least as large as the absolute value of the nth coefficient for any other function in $\mathcal{S}$?

Exercises 11.44.

1. Give an example of a function that is univalent but not analytic in the disk $|z| < 1$.
2. (a) If $f(z) \in \mathcal{S}$, show that for any nonzero complex number t, $|t| \le 1$, the function $f(tz)/t \in \mathcal{S}$.
   (b) If $f(z) = z/(1-z)^2$ and $|t_0| > 1$, show that $f(t_0 z)/t_0 \notin \mathcal{S}$.
3. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is in $\mathcal{S}$, show that, for each integer n, there exists a function $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ in $\mathcal{S}$ such that $b_n = |a_n|$.
4. For α real, verify that $z/(1 - e^{i\alpha} z)^3$ is univalent in $|z| < \frac{1}{2}$, but in no larger disk centered at the origin.
5. If $f(z) \in \mathcal{S}$, show that $z(f(z^k)/z^k)^{1/k} \in \mathcal{S}$ for every positive integer k.
6. Let $f(z)$ be analytic in a domain D and suppose C is a closed contour in D. Prove that $\int_C \overline{f(z)} f'(z)\,dz$ is purely imaginary.
7. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\sum_{n=2}^{\infty} n|a_n| \le 1$, show that $f(z) \in \mathcal{S}$.
8. If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in $\mathcal{S}$, show that $\sum_{n=2}^{\infty} n|a_n| \le 1$.