محاضرة تمهيدى ماجستير

مناقشة الاتى وحل بعض التمارين من كتاب

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Complex Variables with Applications

13

Analytic Continuation

We have previously seen that an analytic function is determined by its behavior at a sequence of points having a limit point. This was precisely the content of the identity theorem (see Theorem 8.48) which is also referred to as the principle of analytic continuation. For example, as a consequence, there is precisely a unique entire function on $\mathbb{C}$ which agrees with $\sin x$ on the real axis, namely $\sin z$. But we have not yet explored the following question: If $f(z)$ is analytic in a domain D_1, is there a function analytic in a different domain D_2 that agrees with $f(z)$ in $D_1 \cap D_2$? Analytic continuation deals with the problem of properly redefining an analytic function so as to extend its domain of analyticity. In the process, we come across functions for which no such extension exists. Finally, we apply our knowledge of analytic continuation to two of the most important functions in analysis, the gamma function and the Riemann-zeta function, defined originally by a definite integral and an infinite series, respectively.

13.1 Basic Concepts

Consider the power series

$$f_0(z) = \sum_{n=0}^{\infty} z^n.$$

This power series converges for $|z| < 1$, and hence, $f_0(z)$ is analytic in the disk $|z| < 1$ and represents there the function $f(z) = 1/(1-z)$. Although the power series diverges at each point on $|z| = 1$, $f(z)$ is analytic in $\mathbb{C} \setminus \{1\}$. For any point $z_0 \neq 1$, the Taylor series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \tag{13.1}$$

is valid when $|z - z_0| < |1 - z_0|$ (see Figure 13.1). The disk in which (13.1)

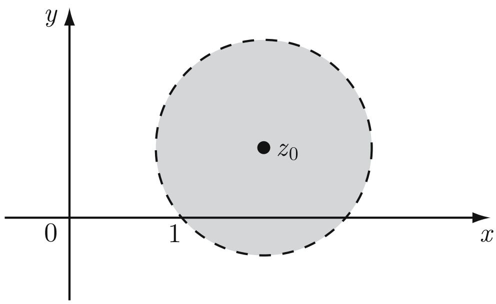

Figure 13.1.

converges may or may not have points in common with the disk $|z| < 1$. For example,

$$f_1(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(e^{i\alpha})}{n!}(z - e^{i\alpha})^n \quad (0 < \alpha < 2\pi)$$

converges in a disk that overlaps $|z| < 1$; but the disk, $|z - 2| < 1$, in which

$$f_2(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(z - 2)^n$$

converges does not. In Figure 13.2, we show the domains in which $f_0(z)$, $f_1(z)$, and $f_2(z)$ converge. In their respective domains of convergence, they all represent the same function $f(z) = 1/(1 - z)$. In addition, the integral

$$\int_0^{\infty} e^{-t(1-z)}\, dt$$

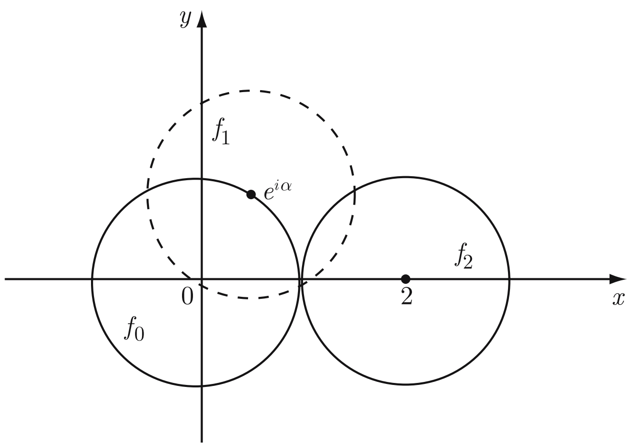

Figure 13.2.

converges for $\mathrm{Re}\, z < 1$ and it can be easily checked that the integral represents $f(z) = 1/(1-z)$ in this half-plane. But they agree with $f_0(z) = \sum_{n=0}^{\infty} z^n$ $(|z| < 1)$ for a certain value of z although they appear different. In fact they agree with $f(z) = 1/(1-z)$ which is analytic for all $z \neq 1$. So we see that apparently unrelated functions may actually represent the same analytic function in different domains.

Suppose $f_0(z)$ is known to be analytic in a domain D_0. We wish to determine the largest domain $D \supset D_0$ for which there exists an analytic function $f(z)$ such that $f(z) \equiv f_0(z)$ in D_0. As we have just seen in the first example, $\mathbb{C} \setminus \{1\}$ is the largest domain containing $|z| < 1$ in which an analytic function may be defined that agrees with $f_0(z) = \sum_{n=0}^{\infty} z^n$ in $|z| < 1$. In our terminology, we say that f_0 has an analytic continuation from the unit disk $|z| < 1$ into the punctured plane $\mathbb{C} \setminus \{1\}$. To see how one can carry out the process of analytic continuations, we need to introduce several definitions.

A function $f(z)$, together with a domain D in which it is analytic, is said to be a *function element* and is denoted by (f, D). Two function elements (f_1, D_1) and (f_2, D_2) are called *direct analytic continuations* of each other iff

$$D_1 \cap D_2 \neq \emptyset \quad \text{and} \quad f_1 = f_2 \quad \text{on } D_1 \cap D_2.$$

Whenever there exists a direct analytic continuation of (f_1, D_1) into a domain D_2, it must be uniquely determined, for any two direct analytic continuations would have to agree on $D_1 \cap D_2$, and by the identity theorem (see Theorem 8.48) would consequently have to agree throughout D_2. That is, given an analytic function f_1 on D_1, there is at most one way to extend f_1 from D_1 into D_2 so that the extended function is analytic in D_2. Thus, one of the main uses of this idea is to extend the functional relations, initially valid for a small domain D_1, to a larger domain D_2. Sometimes such an extension may not be possible. For instance, if D_1 is the punctured unit disk $0 < |z| < 1$ and D_2 is the unit disk, then the function $f_1(z) = 1/z$ cannot be extendable analytically from D_1 into D_2. Similarly, if

$$D_1 = \mathbb{C} \setminus \{z : \mathrm{Re}\, z \leq 0, \, \mathrm{Im}\, z = 0\}, \quad \text{and} \quad D_2 = \mathbb{C},$$

then, for $f_1(z) = \mathrm{Log}\, z$, no extension from D_1 to D_2 is possible.

Remark 13.1. Consider the series

$$f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

This series converges for $|z| \leq 1$ and $f_1(z)$ is analytic in the disk $|z| < 1$, and represents the function

$$f(z) = -\int_0^z \frac{\mathrm{Log}\,(1-t)}{t}\, dt.$$

However, $f_1(z)$ cannot be continued analytically to a domain D with $1 \in D$, since

$$f_1''(z) = \sum_{n=2}^{\infty} \frac{n-1}{n} z^{n-2} \longrightarrow \infty \quad \text{as } z \to 1^+.$$

This observation shows that the convergence or divergence of power series at a point on the circle of convergence does not determine whether the function which defines the series can or cannot be continued along that point. ●

The property of being a direct analytic continuation is not transitive. That is, even if (f_1, D_1) and (f_2, D_2) are direct analytic continuations of each other, and (f_2, D_2) and (f_3, D_3) are direct analytic continuations of each other, we cannot conclude that (f_1, D_1) and (f_3, D_3) are direct analytic continuations of each other. A simple example of this occurs whenever D_1 and D_3 have no points in common. However, there is a relationship between $f_1(z)$ and $f_3(z)$ that is worth exploring.

Suppose $\{(f_1, D_1), (f_2, D_2), \ldots, (f_n, D_n)\}$ is a finite set of function elements with the property that (f_k, D_k) and (f_{k+1}, D_{k+1}) are direct analytic continuations of each other for $k = 1, 2, 3, \ldots, n-1$. Then the set of function elements are said to be *analytic continuations* of one another. Such a set of function elements is then called a *chain*.

Example 13.2. Define (see Figure 13.3)

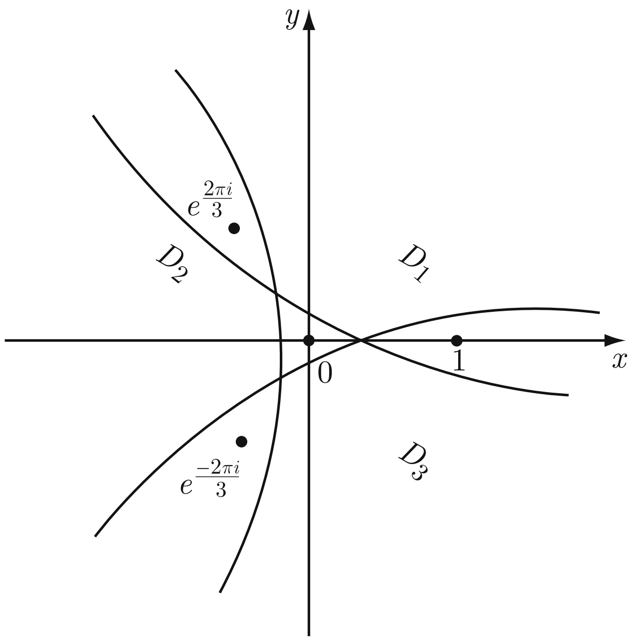

Figure 13.3. Illustration for a chain with $n = 3$

$$f_1(z) = \operatorname{Log} z \quad \text{for } z \in D_1$$
$$f_2(z) = \operatorname{Log} z \quad \text{for } z \in D_2$$
$$f_3(z) = \operatorname{Log} z + 2\pi i \quad \text{for } z \in D_3.$$

Then $\{(f_1, D_1), (f_2, D_2), (f_3, D_3)\}$ is a chain with $n = 3$. Note that $0 = f_1(1) \neq f_3(1) = 2\pi i$. ●

Note that (f_i, D_i) and (f_j, D_j) are analytic continuations of each other if and only if they can be connected by finitely many direct analytic continuations. If $\gamma : [0, 1] \to \mathbb{C}$ is a curve and if there exists a chain $\{(f_i, D_i)\}_{1\leq i\leq n}$, of function elements such that

$$\gamma([0, 1]) \subset \cup_{i=1}^{n} D_i, \; z_0 = \gamma(0) \in D_1, \; z_n = \gamma(1) \in D_n,$$

then we say that the function element (f_n, D_n) is an analytic continuation of (f_1, D_1) along the curve γ. That is a function element (f, D) can be analytically continued along a curve if there is a chain containing (f, D) such that each point on the curve is contained in the domain of some function element of the chain. As another example, the domains of a chain are also shown in Figure 13.4. In some situations, analytic continuation of function element are carried out easily by means of power series. In this case, a chain is a sequence of overlapping disks.

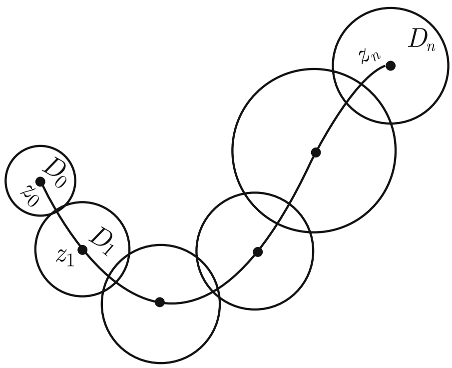

Figure 13.4. Illustration for a chain

Given a chain $\{(f_1, D_1), (f_2, D_2), \ldots, (f_n, D_n)\}$, can a function $f(z)$ be defined such that $f(z)$ is analytic in the domain $\{D_1 \cup D_2 \cup \cdots \cup D_n\}$? Certainly this can be done when $n = 2$. The function

$$f(z) = \begin{cases} f_1(z) & \text{if } z \in D_1 \\ f_2(z) & \text{if } z \in D_2, \end{cases}$$

is analytic in $D_1 \cup D_2$. If $D_1 \cap D_2 \cap \cdots \cap D_n \neq \emptyset$, we can show by induction that f defined by $f(z) = f_i(z)$ for $z \in D_i$ $(i = 1, 2, \ldots, n)$ is analytic. However,

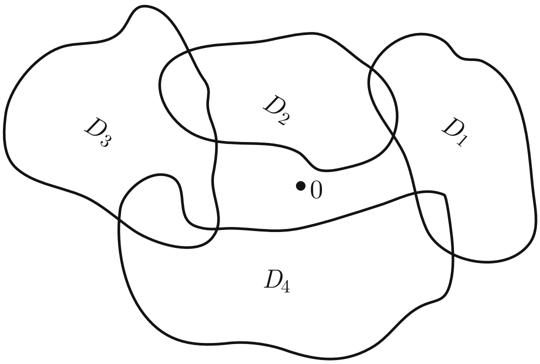

Figure 13.5.

the proof for the general case fails. Consider the four domains illustrated in Figure 13.5. For a fixed branch of $\log z$, set $f_1(z) = \log z$ in D_1. The function element (f_1, D_1) determines a unique direct analytic continuation (f_2, D_2), which determines (f_3, D_3), which determines (f_4, D_4). We thus have the chain $\{(f_1, D_1), (f_2, D_2), (f_3, D_3), (f_4, D_4)\}$. However, in the domain $D_1 \cap D_4$ it is not true that $f_1(z) = f_4(z)$. We actually have $f_4(z) = f_1(z) + 2\pi i$ for all points in $D_1 \cap D_4$. The difference in the two functions lies in the fact that the argument of the multiple-valued logarithmic function has increased by 2π after making a complete revolution around the origin. Note also that we can continue (f_1, D_1) into the domain D_3 by different chains and come up with different functions. For the chains $\{(f_1, D_1), (f_2, D_2), (f_3, D_3)\}$ and $\{(f_1, D_1), (g_1, D_4), (g_2, D_3)\}$, we have the values of f_3 and g_2 differing by $2\pi i$. Before we continue the discussion, let us present our case by a concrete example.

Example 13.3. Consider the function $f(z)$, initially defined on the disk $D = \{z : |z-1| < 1\}$ by the series expansion

$$f(z) = z^{1/2} = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \cdots .$$

Here it is understood that we start with the series representation of the principal branch of $\sqrt{z}$:

$$f(z) = e^{(1/2)\,\mathrm{Log}\, z} = (1 + (z-1))^{1/2}.$$

Note also that f is analytic in D. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed contour given by $\gamma(t) = e^{it}$, starting from $z_0 = \gamma(0) = 1$. Then $f(z)$ actually has an analytic continuation along γ. In fact, we have an explicit convergent power series about e^{it} (write $z^{1/2} = e^{it/2}[1 + (z - e^{it})/e^{it}]^{1/2}$):

$$f_t(z) = e^{it/2} + \frac{1}{2}e^{-it/2}(z - e^{it}) + \frac{1}{2}\left(\frac{1}{2} - 1\right)\frac{1}{2!}e^{-3it/2}(z - e^{it})^2 + \cdots ,$$

where $z \in D_t = \{z : |z - e^{it}| < 1\}$. Thus, after one complete round along the unit circle, we end up at $z = 2\pi$ by

$$f_{2\pi}(z) = -\left[1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \cdots\right]$$

which is just the other branch of $\sqrt{z}$. The initial and final function elements in this case are $(e^{(1/2)\,\mathrm{Log}\,z}, D)$ and $(-e^{(1/2)\,\mathrm{Log}\,z}, D)$, respectively. Also, we observe that the domain formed by the union of all the domains D_t (which can be clearly covered by finitely many such disks), $0 \le t \le 2\pi$, surrounding the origin is not simply connected. In the case of a simply connected domain, the result of the continuation will be unique, no matter what chain is used. This is the substance of the Monodromy Theorem. ●

The difference between single-valued and multiple-valued functions may be viewed from another point of view. Suppose $f(z)$ is analytic in a domain D. A point z_1 is said to be a *regular point* of $f(z)$ if the function element (f, D) can be analytically continued along some curve from a point in D to the point z_1. The set of all regular points of $f(z)$ is called the *domain of regularity* for $f(z)$.

As we have seen, the function $f_0(z) = \sum_{n=0}^{\infty} z^n$ has domain of regularity $\{z : z \neq 1\}$. Note that the function $f(z) = 1/(1-z)$ is analytic in the domain of regularity for $f_0(z)$ and agrees with $f_0(z)$ at all points where they are both analytic.

Consider now the function

$$F_0(z) = \int_0^z f_0(\zeta)\,d\zeta = \int_0^z \left(\sum_{n=0}^{\infty} \zeta^n\right) d\zeta = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} \quad (|z| < 1),$$

where the path of integration lies in the unit disk. The function

$$F(z) = \int_0^z \frac{d\zeta}{1-\zeta} = -\,\mathrm{Log}\,(1-z)$$

agrees with $F_0(z)$ in the disk $|z| < 1$, and is analytic everywhere in the plane except $z = 1$ and the ray $\mathrm{Arg}\,(1-z) = \pi$ (i.e., the ray along the positive real axis beginning at $z = 1$). The function

$$F_1(z) = -\log(1-z) \qquad (0 < \arg(1-z) < 2\pi)$$

is a continuation of $F(z)$ from the half-plane $0 < \mathrm{Arg}\,(1-z) < \pi$ to the whole plane, excluding the point $z = 1$ and the ray $\mathrm{Arg}\,(1-z) = 0$.

Thus the domain of regularity for

$$F_0(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$$

is $\{z : z \neq 1\}$. Note, however, that there does not exist a function that is both analytic in the domain of regularity for $F_0(z)$ and agrees with $F_0(z)$ in the disk $|z| < 1$. As we shall see by the next theorem, this phenomenon occurs only because $\{z : z \neq 1\}$ is a multiply connected domain.

Remark 13.4. We say that the multiple-valued function $\log(1-z)$ is regular in the domain $\{z : z \neq 1\}$ because each such point is a regular point. Some authors allow multiple-valued functions to be analytic. Their definition of analytic then corresponds to our definition of regular. This next theorem shows us that a regular function is always single-valued (hence analytic) in a simply connected domain. ●

Theorem 13.5. (Monodromy Theorem) *Let D be a simply connected domain, and suppose $f_0(z)$ is analytic in a domain $D_0 \subset D$. If the function element (f_0, D_0) can be analytically continued along every curve in D, then there exists a single-valued function $f(z)$ that is analytic throughout D with $f(z) \equiv f_0(z)$ in D_0.*

Proof. We outline the proof, leaving some details for the interested reader. Suppose the conclusion is false. Then there exist points $z_0 \in D_0, z_1 \in D$, and curves C_1, C_2 both having initial point z_0 and terminal point z_1 such that (f_0, D_0) leads to a different function element in a neighborhood of z_1 when analytically continued along C_1 than when analytically continued along C_2 (see Figure 13.6). This means that (f_0, D_0) does not return to the same function element when analytically continued along the closed curve $C_1 - C_2$.

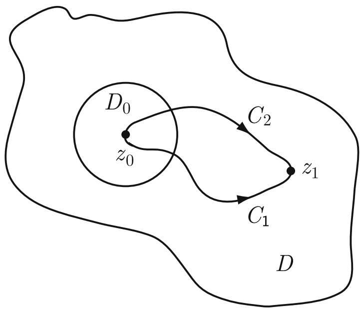

Figure 13.6.

To prove the theorem, it thus suffices to show that the function element (f_0, D_0), $D_0 \subset D$, can be continued along any closed curve lying in D and return to the same value. In the special case that the closed curve C is a rectangle, the proof will resemble that of Theorem 7.39.

Divide the rectangle C into four congruent rectangles, as illustrated in Figure 7.16. Continuation along C produces the same effect as continuation along these four rectangles taken together. If the conclusion is false for C, then it must be false for one of the four sub-rectangles, which we denote by C_1. We then divide C_1 into four congruent rectangles, for one of which the conclusion is false. Continuing the process, we obtain a nested sequence of rectangles for which the conclusion is false. According to Lemma 2.25, there is exactly one point, call it z^*, belonging to all the rectangles in the nest.

Since $z^* \in D$, there exists a function element (f^*, D^*) with $z^* \in D^* \subset D$. For n sufficiently large, the rectangle C_n of the nested sequence is contained in D^*. But this means that $f^*(z)$ is analytic in a domain containing C_n, contrary to the way C_n was defined. This contradiction concludes the proof in the special case in which the curve is a rectangle. For the general proof, see Hille [Hi]. ∎

Suppose $f(z)$ is analytic in a domain D and z_0 is a boundary point of D. The point z_0 will be a regular point of $f(z)$ if, for some disk D_0 centered at z_0, there is a function element (f_0, D_0) such that $f_0(z) \equiv f(z)$ in the domain $D_0 \cap D$. Any boundary point of D that is not a regular point of $f(z)$ is said to be a *singular point* of $f(z)$.

For the function $f(z) = \sum_{n=0}^{\infty} z^n$ $(|z| < 1)$, we have seen that each point on the circle $|z| = 1$ is a regular point except for the point $z = 1$. That all points on the circle cannot be regular is a consequence of the following theorem.

Theorem 13.6. *If the radius of convergence of the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is R, then $f(z)$ has at least one singular point on the circle $|z| = R$.*

Proof. Denote the disk $|z| < R$ by D, and suppose that all points on $|z| = R$ are regular points. Then, for each point z_α on the circle, we can find a function f_α defined in a disk D_α centered at z_α such that the function element (f_α, D_α) is a direct analytic continuation of (f, D). Since $\cup_\alpha D_\alpha$ covers the compact set $|z| = R$, a finite subcover $(D_1, D_2, \ldots, D_n)$ may be found. The function g defined by

$$g(z) = \begin{cases} f(z) & \text{if } z \in D \\ f_i(z) & \text{if } z \in D_i, \end{cases}$$

is analytic in the domain $D' = D \cup D_1 \cup D_2 \cup \cdots \cup D_n$. Since D' contains the disk $|z| \le R$, the domain must also contain the disk $|z| \le R + \epsilon$ for some positive ϵ. Hence the power series representation $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is valid in the disk $|z| < R + \epsilon$, contradicting the fact that the Maclaurin series for $f(z)$ has radius of convergence R. ∎

Corollary 13.7. *If $f(z)$ is analytic in the disk $|z - z_0| < R$ and the Taylor series expansion about $z = z_0$ has radius of convergence R, then $f(z)$ has at least one singular point on the circle $|z - z_0| = R$.*

Proof. Set $\zeta = z - z_0$, and apply the theorem to $f(\zeta)$. ∎

Although we are guaranteed that a power series must have singular points on its circle of convergence, determining their location is, in general, a difficult problem. By placing a restriction on the coefficients, we can locate a particular singular point. Here is one of the results that we have in this direction.

Theorem 13.8. *Suppose* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *has radius of convergence* $R < \infty$. *If* $a_n \geq 0$ *for every* n, *then* $z = R$ *is a singular point of* f.

Proof. If $z = R$ is not a singular point, then $f(z)$ is analytic in some disk $D_0 : |z - R| < \epsilon$. For a positive number $\rho\,(< R)$ sufficiently close to R, we can find an open disk D_1 centered at $z = \rho$ that contains the point $z = R$ and is contained in D_0. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\rho)}{n!}(z-\rho)^n \tag{13.2}$$

converges at a point $z = R + \delta$ $(\delta > 0)$ (see Figure 13.7).

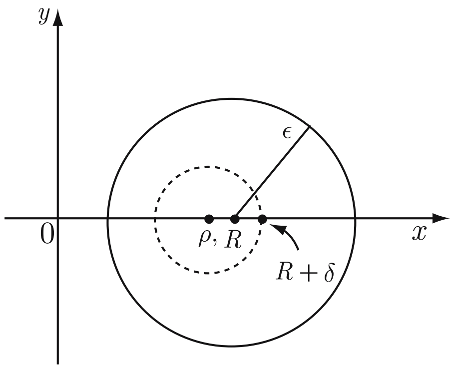

Figure 13.7.

According to Theorem 13.6, the series $\sum_{n=0}^{\infty} a_n z^n$ has a singular point somewhere on the circle $|z| = R$, say $Re^{i\theta_0}$. Hence the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\rho e^{i\theta_0})}{n!}(z-\rho e^{i\theta_0})^n$$

has radius of convergence $R - \rho$ (if the radius of convergence were larger, then $Re^{i\theta_0}$ would not be a singular point). Note that for each n we have

$$f^{(n)}(\rho e^{i\theta_0}) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)a_k(\rho e^{i\theta_0})^{k-n}. \tag{13.3}$$

Since $a_n \geq 0$, we obtain from (13.3) the inequality

$$\left| f^{(n)}(\rho e^{i\theta_0}) \right| \leq \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)a_k\rho^{k-n} = f^{(n)}(\rho).$$

Thus

$$\frac{1}{R-\rho} = \limsup_{n\to\infty} \left| \frac{f^{(n)}(\rho e^{i\theta_0})}{n!} \right|^{1/n} \leq \limsup_{n\to\infty} \left| \frac{f^{(n)}(\rho)}{n!} \right|^{1/n},$$

which means that the radius of convergence of (13.2) is at most $R - \rho$. This contradicts the fact that the series converges at $z = R + \delta$. Therefore, $z = R$ is a singular point of $f(z)$. ■

We have shown that a power series must have at least one singular point on its circle of convergence. The question arises as to whether there is an upper bound on the number of singular points on the circle. We will show that it is possible for every such point to be singular. If $f(z)$ is analytic in a domain whose boundary is C, and every point on C is a singular point of $f(z)$, then C is said to be the *natural boundary* of $f(z)$. In such a case, the domain of regularity is the same as the domain of analyticity.

We will make use of the following lemma in constructing a power series with a natural boundary.

Lemma 13.9. *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence R. If $f(re^{i\theta_0}) \to \infty$ as $r \to R$, then the point $Re^{i\theta_0}$ is a singular point of $f(z)$.*

Proof. If $Re^{i\theta_0}$ is a regular point, then there is a function $g(z)$ that is analytic in a disk centered at $Re^{i\theta_0}$ and agrees with $f(z)$ for $|z| < R$. But then

$$\lim_{r\to R^-} f(re^{i\theta_0}) = \lim_{r\to R^-} g(re^{i\theta_0}) = g(Re^{i\theta_0}),$$

contradicting the fact that the limit on the left side is infinite. ■

Consider now the function

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots,$$

which converges (and so is analytic) in the disk $|z| < 1$. We will show that the circle $|z| = 1$ is a natural boundary for the function $f(z)$. First observe that $f(z) \to \infty$ as $z \to 1$ along the real axis, so that $z = 1$ is a singular point (this is also a consequence of Theorem 13.8). Note that $f(z)$ satisfies the relation $f(z) = z + f(z^2)$. Hence $f(z)$ and $f(z^2)$ simultaneously approach ∞. But then $f(z^2) \to \infty$ when $z^2 \to 1$ through real values, thereby making -1 a singular point. This gives insight into the general method. The function $f(z)$ satisfies the recursive relationship

$$f(z) = z + z^2 + z^4 + \cdots + z^{2^{n-1}} + f(z^{2^n}).$$

For each fixed n, we have

$$|f(z)| \geq \left| f(z^{2^n}) \right| - n \quad (|z| < 1).$$

Since $f(z^{2^n}) \to \infty$ along each ray tending to a 2^n-th root of unity, it follows that each 2^n-th root of unity is a singular point. That is, all points of the form $e^{(2k\pi/2^n)i}$, where k and n are positive integers, are singular points. Now every neighborhood of any other point on the unit circle must contain one of these 2^n-th roots of unity. Hence no point on the unit circle is a regular point. That is, $|z| = 1$ is a natural boundary for $f(z)$.

A similar argument may be used for

$$f(z) = \sum_{n=0}^{\infty} z^{n!},$$

which is analytic in the disk $|z| < 1$. If $z = re^{2\pi(p/q)i}$, where p and q are positive integers and $0 < r < 1$, then (since $e^{2\pi(p/q)n!i} = 1$ for all $n \geq q$) it follows that

$$|f(z)| = \left| \sum_{n=0}^{q-1} r^{n!} e^{2\pi(p/q)n!i} + \sum_{n=q}^{\infty} r^{n!} \right| \geq \sum_{n=q}^{\infty} r^{n!} - q. \tag{13.4}$$

Since the right-hand side of (13.4) tends to ∞ as r tends to 1, all points of the form $e^{2\pi(p/q)i}$ are singular points. But these points are dense on $|z| = 1$, so that the unit circle is a natural boundary for $f(z)$.

Since a power series converges in a disk, its boundary must be a circle. But we have defined natural boundary to include a function for which the domain of analyticity need not be a disk. Consider the function

$$f(z) = \sum_{n=0}^{\infty} e^{-n!z}.$$

Since the series converges uniformly for $\operatorname{Re} z \geq \delta > 0$, the function $f(z)$ is analytic for $\operatorname{Re} z > 0$. We now show that the imaginary axis is a natural boundary for $f(z)$.

Suppose $z = x + 2\pi(p/q)i$, where p is an integer, q is a positive integer, and x is a positive real number. Then

$$|f(z)| = \left| \sum_{n=0}^{q-1} e^{-n!(x+2\pi(p/q)i)} + \sum_{n=q}^{\infty} e^{-n!x} \right| \geq \sum_{n=q}^{\infty} e^{-n!x} - q. \tag{13.5}$$

Because the right side of (13.5) tends to ∞ as x tends to 0, it follows that all points of the form $2\pi(p/q)i$ are singular points. But these points are dense on the imaginary axis so that the imaginary axis furnishes us with a natural boundary for $f(z)$.

Remark 13.10. Let Δ be the unit disk $|z| < 1$ and let $\gamma : [0,1] \to \Delta$ be a curve with $\gamma(0) = 0$ and D be such that $0 \in D \subsetneq \Delta$. Then there is always an analytic continuation of $(\sum_{n=0}^{\infty} z^{n!}, \Delta)$ along γ. However, if $\gamma_1 : [0,1] \to \mathbb{C}$

is given by $\gamma_1(t) = 2it$, there is no analytic continuation of $(\sum_{n=0}^{\infty} z^{n!}, \Delta)$ along γ_1.

Similar comments apply for the function element $(\sum_{n=0}^{\infty} z^{2^n}, \Delta)$. ●

Questions 13.11.

1. If $f(z) = z$ in a domain D_0, can $f(z)$ be analytic in a domain D_1 even though $f(z) \neq z$ in D_1?
2. Can two functions, analytic in the disk $|z| < 1$, agree at infinitely many points there and not agree everywhere in the disk?
3. Can an analytic continuation always be transformed into a direct analytic continuation?
4. Is it possible that the function elements (f, D_1) and (g, D_2) can be connected by an infinite chain of function elements, but by no finite subchain?
5. Why is the domain of regularity a domain?
6. What is the difference between a singular point and a singularity? A regular point and a point of analyticity?
7. Can infinitely many points on the boundary C of a domain be singular without C being a natural boundary?
8. If $D_1, D_2, \dots, D_n$ are domains, when is their union a domain?
9. Is the converse of Lemma 13.9 true?
10. Is there a relationship between gaps in the coefficients of the Maclaurin series for $f(z)$ and the circle of convergence being a natural boundary?
11. Is there a relationship between the Cauchy Theorem and the Monodromy Theorem?
12. What does the Monodromy theorem tell us about $\log z$? About $\sqrt{z}$?

Exercises 13.12.

1. Given a set of real numbers $0 \le \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$, construct a function $f(z)$ such that
   (i) $f(z)$ is analytic in $|z| < 1$;
   (ii) the only singular points of $f(z)$ on the unit circle are at $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}$.
2. Given (f_1, D_1), where $f_1(z) = \sum_{n=0}^{\infty} z^n$ and $D_1 = \{|z| < 1\}$, construct a chain $\{(f_1, D_1), (f_2, D_2), \dots, (f_n, D_n)\}$.
3. Show that the set of regular points of an analytic function is open, and the set of singular points is closed.
4. (a) Show that $f(z) = \sum_{n=0}^{\infty} [z^{2^{n+1}} / (1 - z^{2^{n+1}})]$ is analytic in the domain $|z| < 1$ and the domain $|z| > 1$, and that $|z| = 1$ is a natural boundary for the function in each domain.
   (b) Determine $f(z)$ in each of these domains in closed form.
5. Show that $|z| = 1$ is a natural boundary for $\sum_{n=0}^{\infty} z^{3^n}$.
6. Suppose $\sum_{n=0}^{\infty} a_n z^{n!}$ $(a_n > 0)$ has radius of convergence R. Show that $|z| = R$ is a natural boundary.

7. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic for $|z| < 1$ and that a_n is real for each n. If $\sum_{n=1}^{k} a_n \to \infty$ as $k \to \infty$, show that $z = 1$ is a singular point for $f(z)$.
8. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1 and that the only singularities on the circle $|z| = 1$ are simple poles. Show that the sequence $\{a_n\}$ is bounded.
9. Show that $f(z) = \int_0^1 (1-tz)^{-1}\,dt$ is an analytic continuation of $f_0(z) = \sum_{n=1}^{\infty} z^{n-1}/n$ from the unit disk $|z| < 1$ into the whole complex plane minus the interval $[1, \infty)$.
10. Suppose $f(z) = \sum_{n=0}^{\infty} (-1)^n a_n z^n$ has radius of convergence R and $a_n \geq 0$ for every n. Show that $z = -R$ is a singular point.

13.2 Special Functions

There are functions which arise so frequently in complex analysis that they have intrinsic interest. The gamma function of Euler and the zeta function of Riemann are two such "special functions" which require special attention. As we have seen in the previous chapter, the gamma function is meromorphic with simple poles at $0, -1, -2, \ldots$, and it is free of zeros. Its reciprocal is an entire function, with a simple zero at each nonpositive integers and with no other zeros. This may be expressed as

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}, \tag{13.6}$$

where

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right).$$

Thus we may rewrite (13.6) as

$$\begin{aligned} \frac{1}{\Gamma(z)} &= \left[\lim_{n\to\infty} z e^{[1+(1/2)+(1/3)+\cdots+(1/n)]z - z\ln n} \right] \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{z+k}{k} \right) e^{-z/k} \\ &= \lim_{n\to\infty} \left[z e^{-z\ln n} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) \right] \\ &= \lim_{n\to\infty} \frac{z(z+1)(z+2)\cdots(z+n)}{n^z n!}. \end{aligned}$$

This leads to an alternate expression for the gamma function, namely

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}, \tag{13.7}$$

which is defined for all values except zero and the negative integers. Equation (13.7) is referred to as "Gauss's formula". Therefore, for all values of z with $z \neq 0, -1, -2, \dots$, we get that

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{nz}{z+n+1} \left(\frac{n!n^z}{z(z+1)\cdots(z+n)} \right) = z\Gamma(z).$$

In this way, we obtain an alternate proof of the functional equation of the gamma function shown in the previous chapter. There is still one more method to obtain this equation as we shall see soon.

In real analysis, the gamma function is defined in terms of the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad (x > 0). \tag{13.8}$$

Note that the integral (13.8) makes no sense when $x \leq 0$. Indeed, as $e^{-t} > e^{-1}$ for all $t \in (0,1)$, and for $0 < \delta < 1$

$$\int_\delta^1 t^{x-1} e^{-t}\, dt \geq \frac{1}{e} \int_\delta^1 t^{x-1}\, dt = \frac{1}{e} \left(\frac{1-\delta^x}{x} \right)$$

which approaches ∞ as $\delta \to 0^+$ for $x < 0$. Thus, the improper integral (13.8) diverges for $x < 0$. It is easy to see that it also diverges at $x = 0$.

To see that the integral (13.8) converges for all positive x, we write

$$\Gamma(x) = \int_0^1 t^{x-1} e^{-t}\, dt + \int_1^\infty t^{x-1} e^{-t}\, dt = I_1 + I_2.$$

Since $e^{-t} \leq 1$ for $t \geq 0$, it follows that the integral (13.8) converges at $t = 0$ because for each $\delta > 0$,

$$\int_\delta^1 t^{x-1} e^{-t}\, dt \leq \int_\delta^1 t^{x-1}\, dt = \frac{1-\delta^x}{x} < \frac{1}{x}$$

so that $I_1 \leq 1/x$. For large t,

$$t^{x-1} e^{-t} \leq e^{t/2} e^{-t} = e^{-t/2}$$

so that the integral converges at ∞. In fact, since $\lim_{t\to\infty}(t^{x-1}/e^t) = 0$, the integrand of I_2 is also bounded so that

$$\int_N^\infty t^{x-1} e^{-t}\, dt \leq \int_N^\infty \frac{t^{x-1}}{t^{x+1}}\, dt = \frac{1}{N} \quad (N \geq N(x)).$$

Hence $\Gamma(x)$ is defined for all $x > 0$. An integration by parts gives

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt = -\frac{t^x}{e^t}\bigg|_0^\infty + x \int_0^\infty t^{x-1} e^{-t}\, dt = x\Gamma(x). \tag{13.9}$$

Note that (13.6) has been shown to satisfy $\Gamma(x+1) = x\Gamma(x)$ for complex values of x. From (13.9) and the fact

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1,$$

it follows that $\Gamma(n+1) = n!$ for all positive integers n.

Consider now the complex-valued function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt. \tag{13.10}$$

For $z = x+iy, x > 0$, we have

$$\left|t^{z-1}\right| = \left|e^{(x-1)\operatorname{Log} t + iy \operatorname{Log} t}\right| = e^{(x-1)\operatorname{Log} t} = t^{x-1}.$$

Hence the integral (13.10) converges absolutely for $x > 0$, with

$$|\Gamma(z)| \le \int_0^\infty \left|t^{z-1}e^{-t}\right|\,dt = \Gamma(x)$$

so that (13.10) is well defined in the half-plane $\operatorname{Re} z > 0$. We wish to show that (13.10) has two important properties: first, it is analytic for $\operatorname{Re} z > 0$; second, it agrees with (13.7) for $\operatorname{Re} z > 0$. This will justify the apparently inexcusable notation in which the same letter is used for (13.10) and (13.7).

Let K be a compact subset of the half-plane $\operatorname{Re} z > 0$. For $z = x+iy \in K$, choose x_0, x_1 so that $0 < x_0 \le x \le x_1 < \infty$. Then, we have

$$|\Gamma(z)| \le \Gamma(x) \le \int_0^1 t^{x_0-1}e^{-t}\,dt + \int_1^\infty t^{x_1-1}e^{-t}\,dt < \Gamma(x_0) + \Gamma(x_1).$$

Thus $\Gamma(z)$ is bounded in the infinite strip

$$x_0 \le \operatorname{Re} z \le x_1. \tag{13.11}$$

For $n \ge 1$, we set

$$\Gamma_n(z) = \int_{1/n}^n t^{z-1}e^{-t}\,dt.$$

We will show that $\Gamma_n(z)$ is analytic for $\operatorname{Re} z > 0$, with

$$\Gamma_n'(z) = \int_{1/n}^n t^{z-1}e^{-t}\ln t\,dt.$$

To this end, we show that, on any strip of the form (13.11), the expression

$$\begin{aligned}\left|\frac{\Gamma_n(z+h)-\Gamma_n(z)}{h} - \int_{1/n}^n t^{z-1}e^{-t}\ln t\,dt\right| &= \left|\int_{1/n}^n t^{z-1}e^{-t}\left(\frac{t^h-1}{h} - \ln t\right)dt\right| \\ &\le \int_{1/n}^n t^{x-1}e^{-t}\left|\frac{t^h-1}{h} - \ln t\right|dt\end{aligned}$$

can be made arbitrarily small for $|h|$ sufficiently small. Using the mean-value theorem and the uniform continuity of $\ln t$ on the interval $[1/n, n]$, we can show that $(t^h - 1)/h$ converges uniformly to $\ln t$ for $1/n \le t \le n$. It thus follows when $|h| < \delta(\epsilon)$ that the last integral above is bounded above by

$$\epsilon \int_{1/n}^{n} t^{x-1} e^{-t}\, dt < \epsilon \Gamma(x) < \epsilon(\Gamma(x_0) + \Gamma(x_1)).$$

Hence $\Gamma_n(z)$ is analytic (for $x_0 < \operatorname{Re} z < x_1$), with

$$\Gamma_n'(z) = \int_{1/n}^{n} t^{z-1} e^{-t} \ln t\, dt.$$

But

$$\lim_{n\to\infty} \Gamma_n(z) = \Gamma(z)$$

for $x_0 \le \operatorname{Re} z \le x_1$. Since $\Gamma_n(z)$ is locally uniformly bounded in the right half-plane, Montel's theorem (Theorem 11.14) may be applied to show that $\Gamma(z)$ is analytic for $\operatorname{Re} z > 0$.

We now show that the integral definition (13.10) agrees with (13.7) for $x = \operatorname{Re} z > 0$. Set

$$\Gamma_n^*(x) = \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt \quad (x > 0,\ n \ge 1).$$

Integrating by parts, we obtain

$$\begin{aligned} \Gamma_n^*(x) &= \frac{t^x}{x}\left(1 - \frac{t}{n}\right)^n \Bigg|_0^n + \frac{1}{x}\int_0^n t^x \left(1 - \frac{t}{n}\right)^{n-1} dt \\ &= \frac{1}{x}\int_0^n t^x \left(1 - \frac{t}{n}\right)^{n-1} dt. \end{aligned}$$

Integrating by parts $n-1$ more times, we get

$$\begin{aligned} \Gamma_n^*(x) &= \frac{1}{x}\, \frac{n-1}{n(x+1)}\, \frac{n-2}{n(x+2)} \cdots \frac{1}{n(x+n-1)} \times \int_0^n t^{x+n-1}\, dt \\ &= \frac{(n-1)!n^{x+n}}{n^{n-1}x(x+1)\cdots(x+n)} \\ &= \frac{n!n^x}{x(x+1)\cdots(x+n)}. \end{aligned}$$

Thus for $x > 0$,

$$\lim_{n\to\infty} \Gamma_n^*(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}. \tag{13.12}$$

If we can now show that

$$\lim_{n\to\infty} \Gamma_n^*(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

on the interval $[1,2]$, it will then follow from the identity theorem that

$$\lim_{n\to\infty} \frac{n!n^z}{z(z+1)\,\cdots\,(z+n)} = \int_0^\infty t^{z-1}e^{-t}\,dt$$

in the largest domain containing the interval $[1,2]$ in which both functions are analytic; that is, the representations (13.7) and (13.10) will have been shown to be equal in the right half-plane.

For $n > N$, we have

$$\Gamma_n^*(x) > \int_0^N t^{x-1}\left(1-\frac{t}{n}\right)^n\,dt \quad (1 \le x \le 2). \tag{13.13}$$

The sequence of polynomials $f_n(t) = (1-t/n)^n$ converges uniformly to e^{-t} on any finite interval $[a,b]$. Furthermore,

$$f_n(t) \le f_{n+1}(t) \le e^{-t}$$

for n sufficiently large. Hence for each fixed x, the integrand of (13.13) (as a function of t) converges uniformly to $t^{x-1}e^{-t}$ on the interval $[0,N]$. Therefore,

$$\lim_{n\to\infty} \Gamma_n^*(x) \ge \lim_{n\to\infty} \int_0^N t^{x-1}\left(1-\frac{t}{n}\right)^n\,dt = \int_0^N t^{x-1}e^{-t}\,dt.$$

Since N is arbitrary, it follows that

$$\lim_{n\to\infty} \Gamma_n^*(x) \ge \int_0^\infty t^{x-1}e^{-t}\,dt. \tag{13.14}$$

But $\Gamma_n^*(x) \le \int_0^n t^{x-1}e^{-t}\,dt \le \int_0^\infty t^{x-1}e^{-t}\,dt$, so that

$$\lim_{n\to\infty} \Gamma_n^*(x) \le \int_0^\infty t^{x-1}e^{-t}\,dt. \tag{13.15}$$

Combining (13.14) and (13.15), we see that (13.10) agrees with (13.7) for $1 \le x \le 2$, and consequently they must agree in the right half-plane. Hence (13.7) (or (13.6)) may be viewed as a direct analytic continuation of the function

$$\int_0^\infty t^{z-1}e^{-t}\,dt$$

from the domain $\operatorname{Re} z > 0$ to $\mathbb{C}\setminus\{0,-1,-2,\ldots\}$.

Our next discussion concerns the function

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \tag{13.16}$$

known as the *Riemann-zeta function.* (Here we use the traditional notation denoting the complex variable $s = \sigma + it$ rather than $z = x + iy$.) This is one of the most challenging and fascinating functions which has a natural link connecting the set of prime numbers with analytic number theory. We have already met this series at $s = 2$ and $s = 4$ with (p. 433 and Exercise 12.30(6))

$$\zeta(2) = \pi^2/6 \quad \text{and} \quad \zeta(4) = \pi^4/90.$$

Since $f_n(s) = n^{-s} = e^{-s \operatorname{Log} n}$ is an entire function and for $s = \sigma + it$,

$$|n^{-s}| = e^{-\sigma \operatorname{Log} n} = n^{-\sigma},$$

we see that the series (13.16) converges absolutely for $\operatorname{Re} s > 1$ and uniformly for $\operatorname{Re} s \geq \sigma_0 > 1$. Hence $\zeta(s)$ represents an analytic function in the half-plane $\operatorname{Re} s > 1$. Consequently,

$$\zeta'(s) = \sum_{n=1}^{\infty} f_n'(s) = -\sum_{n=2}^{\infty} (\ln n) n^{-s} \quad \text{for } \operatorname{Re} s > 1,$$

and more generally,

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} (\ln n)^k n^{-s} \quad \text{for } \operatorname{Re} s > 1.$$

Now, to see its link with the collection of prime numbers, we prove the following

Theorem 13.13. (Euler's Product Formula) *For $\sigma > 1$, the infinite product $\prod_p (1 - p^{-s})$ converges and*

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right), \tag{13.17}$$

where the product is taken over the set $P = \{2, 3, 5, 7, 11, \dots\}$ of all prime numbers p.

Proof. Since the series $\sum p^{-s}$ converges absolutely for all $\operatorname{Re} s > 1$, and it converges uniformly on every compact subset of the half-plane $\operatorname{Re} s > 1$, the infinite product (13.17) converges. Next we note that for $\sigma > 1$

$$\zeta(s)\frac{1}{2^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots$$

so that

$$\zeta(s)\left(1 - \frac{1}{2^s}\right) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots .$$

Similarly, one can find that

2. Show that the gamma function may be expressed as

$$\Gamma(z) = \int_1^{\infty} e^{-t} t^{z-1}\, dt + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)}.$$

3. Show that $\operatorname{Re}\zeta(s) > 0$ when $\operatorname{Re} s \geq 2$.
4. Show that $(1-1/2^{s-1})\zeta(s)$ is an entire function and may be represented as $\sum_{n=1}^{\infty}(-1)^{n+1}/n^s$ for $\operatorname{Re} s > 1$. Where else does this series converge?

5. For $0 < \operatorname{Re} s < 1$, show that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) dt.$$

6. Show that $\zeta(1-s) = (1/2^{s-1}\pi^s)\cos(\pi s/2)\Gamma(s)\zeta(s)$.
7. Determine an analytic continuation of $\sum_{n=1}^{\infty} z^n/n^{1/4}$.
8. Consider the analytic function

$$f(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n \quad (c > -1).$$

Determine the largest domain to which f can be analytically continued? Determine an analytic continuation of f from the unit disk to a larger domain?