Partial Differential Equations

Partial differential equations (PDE's) arise in connection with various physical and geometrical problems when the functions involved depend on two or more independent variables. It is fair to say that only the simplest physical systems can be modeled by ordinary differential equations (ODE's). Fluid and solid mechanics heat transfer, electromagnetic theory and other areas of physics are full problems that must be modeled by PDE's. The range of application of the PDE's is numerous, compared to that of ODE's. The independent variables involved may be time and or several coordinates in space.

1- Basic Concepts:-

An equation involving one or more partial derivatives of an (unknown) function of two or more independent variables is called partial differential equation. The order of the highest derivative is called the order of the equation. Just as in the case of ordinary differential equation, we say that a partial differential equation is linear, if it is of the first degree in the dependent variable (the unknown function) and its partial derivatives. If each term of such equation contains either the dependent variable or one of its derivatives, the equation is said to be homogeneous; Otherwise it is said to be non-homogeneous.

A solution of a PDE in some region R of the space of the independent variables is a function that has all the partial derivatives appearing in the equation in some domain containing R and satisfies the equation everywhere in R. (Often one merely requires that the equation is continuous on the boundary of R, has those derivative in the interior of R and satisfies the equation in the interior of R)

In general the totality of solutions of partial differential equation is very large. For example the functions

$$u = x^2 - y^2 \qquad u = e^x \cos y \qquad u = \ln(x^2+y^2)$$

which are entirely different from each other are solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

We shall see later that the unique solution of a partial differential equation corresponding to a given physical problem will be obtained by the use of additional information arising from the physical situation. For example, in some cases, the values of the required solution of the problem one the boundary of some domain will be given ("boundary conditions"); in cases when time t is one of the variables, the values of the solution at $t=0$ will be prescribed ("initial conditions")

2- Fundamental Theorem (superposition principle).

If u_1 and u_2 are any solutions of a linear homogeneous PDE in some region, then

$$u = c_1 u_1 + c_2 u_2$$

is also a solution of that equation in that region, where c_1 and c_2 are arbitrary constants.

3- Classification of PDE's:

The general form of the second order PDE can be written as

$$A u_{\xi\xi} + B u_{\xi\eta} + C u_{\eta\eta} + D u_\xi + E u_\eta + F u + G = 0$$

where A, B, C, and so forth, are in general functions in ξ and η. Such an equation can be classified according to the sign of $\Delta = B^2 - 4AC$ and the equation is called

i) elliptic if $\Delta < 0$
ii) parabolic if $\Delta = 0$
iii) hyperbolic if $\Delta > 0$

Since A, B and C are functions of ξ and η (not of u), the classification varies from point to point.

Examples

Classify the following equations:

i) $u_{tt} = c^2 u_{xx}$ one-dimensional wave equation

ii) $u_t = c^2 u_{xx}$ one-dimensional heat equation

iii) $u_{xx} + u_{yy} = 0$ two-dimensional Laplace equation

iv) $u_{xx} + u_{yy} + 2u_x + u = 0$

v) $yU_{xx} - 4U_{xy} + xU_{yy} - 3U_x + 2U_y + U = 0$

Solution

i) $A = 1$ $B = 0$ $C = -c^2$

$\Delta = 4c^2$ and the equation is hyperbolic

ii) $A = c^2$ $B = 0$ $C = 0$

$\Delta = 0$ and the equation is parabolic

iii) $A = 1$ $B = 0$ $C = 1$

$\Delta = -4$ and the equation is elliptic

iv) $A = 1$ $B = 0$ $C = 1$

$\Delta = -4$ and the equation is elliptic

v) $A = y$ $B = -4$ $C = x$

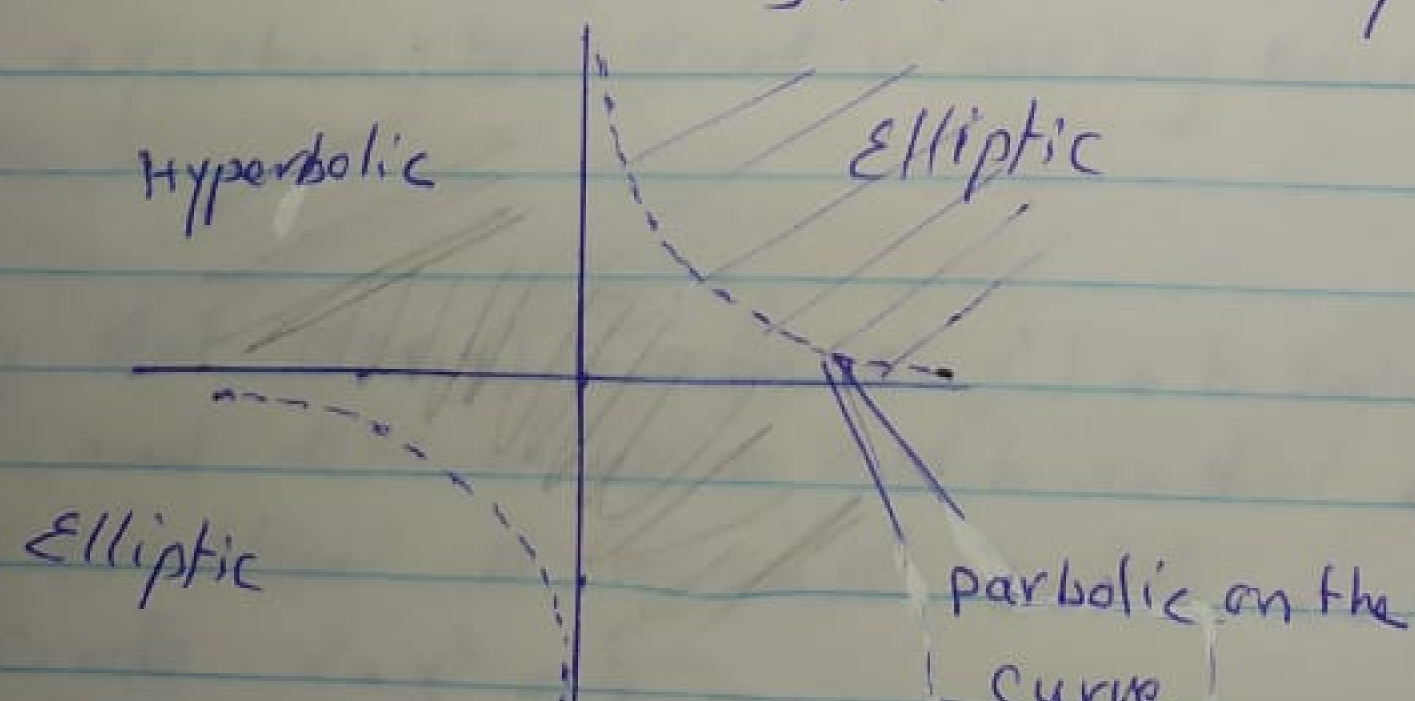
$\Delta = 16 - 4xy$

depends on x and y and we have the following cases

a) if $\Delta = 16 - 4xy = 0$ $xy = 4$ parabolic

b) if $\Delta = 16 - 4xy > 0$ $xy < 4$ hyperbolic

c) if $\Delta = 16 - 4xy < 0$ $xy > 4$ elliptic.



Example

show that

i) $u = e^x \sin y$ and ii) $u = e^x \cos y$

are solutions of the Laplace's equation

$$u_{xx} + u_{yy} = 0$$

i

$u_{xx} = e^x \sin y$

$u_{yy} = -e^x \sin y$

$u_{xx} + u_{yy} = 0$

ii

$u_{xx} = e^x \cos y$

$u_{yy} = -e^x \cos y$

$u_{xx} + u_{yy} = 0$

problems

i) Verify that the following are solutions of the wave equation

a) $u = x^2 + c^2t^2$

b) $u = \cos(ct)\sin(x)$

c) $u = \sin(\omega t)\sin(\frac{w}{c}x)$

ii) verify that the following are solutions of the heat eqn:

a) $u = \exp(-t)\cos x$

b) $u = \exp(-\omega^2c^2t)\sin\omega x$

c) $u = \exp(-c^2t)\sin\omega x$

iii) show that $u(x,t) = V(x+ct) + W(x-ct)$ is a solution of the wave equation; here V and W are any twice differentiable functions

iv) Verify that $u(x,y) = a\ln(x^2+y^2) + b$ satisfies Laplace's equation and determine a and b so that u satisfies the boundary conditions $u = 0$ on the circle $x^2+y^2=1$ and $u = 3$ on the circle $x^2+y^2=4$

v) Show that

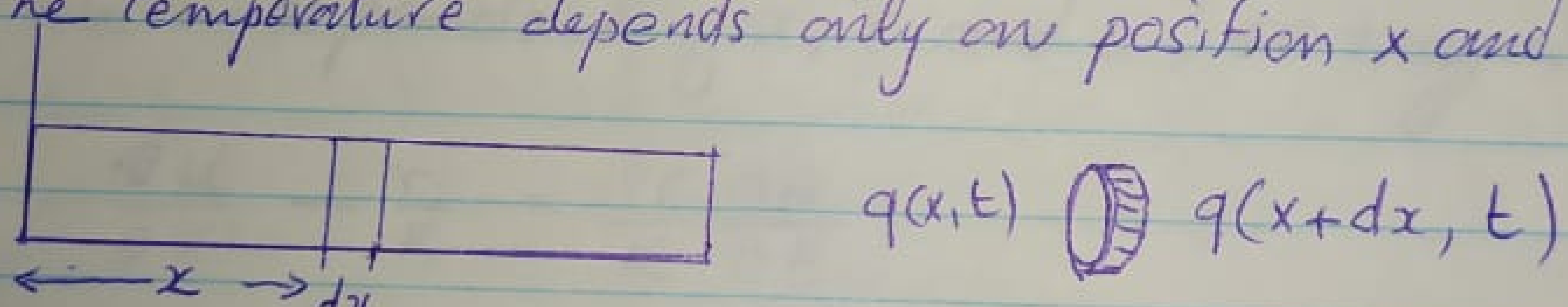
$$u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$$ is a solution of Laplace's equation $\nabla^2 u = 0$

The Heat Equation

Derivation

As a first example of the derivation of a partial differential equation, we consider the problem describing the temperature in a rod of conducting material. In order to simplify the problem as much as possible, we assume that

i) The rod has a uniform cross section
ii) The temperature does not vary from point to a point on a section
iii) The temperature depends only on position x and time t



where $q(x,t)$ is the rate of heat flow at point x and time t

A: area of the section
ρ is the density
c is the heat capacity per unit mass
k " the thermal conductivity
$u(x,t)$ is the temperature and
r is the rate of heat generation per unit volume

We now quantify the law of conservation of energy of the slice of the rod in the form

$$A q(x,t) + A \Delta x r = A q(x+\Delta x, t) + A \Delta x \rho c \frac{\partial u}{\partial t}$$

After some algebraic manipulation, we have

$$[q(x,t) - q(x+\Delta x, t)]/\Delta x + r = \rho c \, \partial u / \partial t$$

but

$$[q(x,t) - q(x+\Delta x, t)]/\Delta x \rightarrow -\frac{\partial q}{\partial x} \quad \text{as } \Delta x \rightarrow 0$$

The limit process thus leaves the law of conservation of energy in the form

$$-\frac{\partial q}{\partial x} + r = \rho c \frac{\partial u}{\partial t}$$

but according to the Fourier's law

$$q = -K \frac{\partial u}{\partial t}$$

Thus the heat balance equation yields

$$\frac{\partial}{\partial x}\left(K \frac{\partial u}{\partial x}\right) + r = \rho c \frac{\partial u}{\partial t}$$

Note that K, ρ and c may be all functions. If, however, they are independent of x, t and u, we may write

$$\frac{\partial^2 u}{\partial x^2} + \frac{r}{K} = \frac{\rho c}{K} \frac{\partial u}{\partial t}$$

This equation is applicable in the regim $0 < x < L$ and for $t > 0$. The quantity $\frac{K}{\rho c}$ is often written as k and is called the diffusivity. For some time we will be working with the heat equation without generation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t} \qquad 0 < x < L \qquad t > 0$$

which, to review, is supposed to describe the temperature u in the rod of length a with uniform properties and cross-section, in which no heat is generated and whose cylindrical surface is insulated

Example
Each of the functions

$$u(x,t) = x^2 + 2xt$$
$$u(x,t) = e^{-kt} \sin(x)$$

satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

and so their sum and difference. Clearly this is not a satisfactory situation either from the mathematical or physical point of view. We woud like the temperature to be uniquely determined. More conditions must be applied on the function u. The appropriate additional conditions are those the describe

i) the initial temperature in the rod, and
ii) what happening at the ends of the rod.

The initial condition is described mathematically as

$$u(x,0) = f(x) \qquad 0 \le x < L$$

where $f(x)$ is a given function of x alone.

Types of Boundary Conditions:

1- Dirichlet condition or "condition of the first kind";

$$u(0,t) = T_0 \qquad u(L,t) = T_1 \qquad t > 0 \quad \text{or}$$
$u(x_0,t) = u(t)$ where x_0 symbolizes an endpoint

ii Neumann condition or "condition of the second kind"

$$\frac{\partial u}{\partial x}(x_0,t) = \beta(t)$$

iii Robin's condition or "condition of the third kind" iv Mixed condition as

$$u(0,t) = u(L,t) \quad \text{or}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}(x,t)\right|_{x=L}$$

Solution of the heat Equation:

Solve the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t} \qquad 0<x<L \quad t>0 \qquad (1)$$

with the conditions
$$u(0,t) = T_0 \qquad (2)$$
$$u(L,t) = T_1 \qquad (3)$$
$$u(x,0) = f(x) \qquad (4)$$

After a long time under the conditions, the variation of temperature with time dies away, In terms of the function $u(x,t)$ that represents the temperature, we thus expect that the limit of $u(x,t)$, as t tends to infinity, exists and depends only on x

$$\lim_{t\to\infty} u(x,t) = V(x)$$

$$\lim_{t\to 0} \frac{\partial u}{\partial t} = 0$$

The function $V(x)$, called the steady state temperature distribution, must still satisfy the boundary conditions and the heat equation, which are valid for all $t>0$. Therefore $V(x)$ should be the solution to the problem

$$\frac{\partial^2 V}{\partial x^2} = 0 \qquad 0<x<L \qquad (5)$$

$$V(0) = T_0 \quad \text{and} \quad V(L) = T_1 \qquad (6)$$

On integrating the differential equation twice, we find

$$V(x) = Ax + B \qquad V(0) = B = T_0$$

$$V(L) = AL + B = AL + T_0 = T_1$$

$$A = \frac{T_1 - T_0}{L}$$

The steady state distribution becomes

$$V(x) = T_0 + (T_1 - T_0)\frac{x}{L} \qquad (7)$$

We now isolate the "rest" of the unknown temperature $u(x,t)$ of the unknown temperature $u(x,t)$ by defining the transient temperature distribution

$$W(x,t) = u(x,t) - V(x)$$

In general the transient also satisfies an initial boundary value problem that is similar to the original one but is distinguished by having a homogeneous partial differential equation and boundary conditions. To illustrate this point we shall treat the problem stated in conditions (1) – (4).

To illustrate this point, we shall treat the problem stated in equations (1) – (4) whose steady state solution is given by equation (7)

$$u(x,t) = W(x,t) + V(x)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 W}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial W}{\partial t} \qquad u(0,t) = T_0 \Rightarrow W(0,t) + V(0) = W(0,T) + T_0$$

$$u(L,t) = T_1 \Rightarrow W(L,t) + V(L) = W(L,t) + T_1$$

Substituting in equation (1) – (4), we get the following initial value problem for W

$$\frac{\partial^2 W}{\partial x^2} = \frac{1}{k}\frac{\partial W}{\partial t} \qquad 0 < x < L, \quad t > 0$$

$$W(0,t) = 0 \quad W(L,t) = 0 \quad W(x,0) = f(x) - V(x) = f(x) - \left[T_0 + \frac{T_1 - T_0}{L}x\right]$$

$$= g(x)$$

The problem in W can be attacked by a method called "product method", "separation of variables" or "Fourier method". For this method to work, it is essential to have homogeneous boundary conditions. The general idea of this method is to assume that the solution of the partial differential equation has the form of a product

$$W(x,t) = \Phi(x)T(t)$$

Since each factor depends on only one variable, we have

$$\frac{\partial^2 W}{\partial x^2} = \Phi''(x)T(t) \qquad \frac{\partial W}{\partial t} = \Phi(x)T'(t)$$

where the dash stands for the derivative w.r.t its corresponding argument.

Thus on substituting in the heat equation and dividing on $\Phi(x)T(t)$, we get

$$\frac{\Phi''}{\Phi} = \frac{1}{K}\frac{T'}{T}$$

The left hand side depends only on x and the right hand side depends only on t. If this equation is to hold for $0<x<L$ and $t>0$, the mutual value of these two functions must be constant.

$$\frac{\Phi''}{\Phi} = \frac{T'}{KT} = p$$

i.e

$$\Phi'' - p\Phi = 0 \quad \text{and} \quad T' - KpT = 0 \qquad (8)$$

The two boundary conditions on W may also be stated in the product form

$$W(0,t) = \Phi(0)T(t) \quad \text{and} \quad W(L,t) = \Phi(L)T(t)$$

There are two ways that these equations can be satisfied for all $t > 0$. Either the function $T(t) = 0$ for all t, or the other factors must be zero. But in the first case $w(x,t) = \phi(x) T(t)$ is also identically zero, leading us back to the trivial solution, therefore we take the other alternative and choose

$$\phi(0) = 0 \qquad \phi(L) = 0 \qquad (9)$$

Our job now is to solve the equations (8) and satisfy the boundary conditions (9) while avoiding the trivial solution.

i) Assume $p > 0$:

$$\phi(x) = a e^{\sqrt{p}x} + b e^{-\sqrt{p}x} \qquad T(t) = e^{pkt}$$

Applying the boundary conditions (9) to $\phi(x)$ leads us to conclude first that $a = 0$ and $b = 0$ so that $\phi(x) = 0$. This gives the trivial solution $W(x,t) = 0$ which is unsatisfactory. The same conclusion is reached we try $p = 0$

ii) Assume $p < 0$

we now try a negative constant. Replacing p by $-\lambda^2$ in equation (8) gives us the two equations

$$\phi'' + \lambda^2 \phi = 0 \qquad T' + \lambda^2 k T = 0$$

whose solutions are

$$\phi(x) = a\cos(\lambda x) + b\sin(\lambda x)$$

$$T(t) = c\exp - \lambda^2 k t$$

$$\phi(0) = a = 0 \quad \text{and} \quad \phi(L) = b\sin\lambda L = 0$$

then $\lambda_n L = n\pi \qquad \lambda_n = \frac{n\pi}{L} \qquad n = 1, 2$

so $T_n = \exp - \lambda_n^2 k t \qquad W_n = \sin \lambda_n x \, e^{-\lambda_n^2 k t}$

we now apply the principle of superposition and form a linear combination of the W's

$$W(x,t) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x \exp -\lambda_n^2 kt$$

$$W(x,0) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x = \sum_{n=1} b_n$$

$$b_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Example 2

Suppose the original problem to be

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad 0 < x < L \qquad t > 0$$

$u(0,t) = T_0 \quad t > 0$

$u(L,t) = T_1 \quad t > 0$

$u(x,0) = 0 \quad 0 < x < L$

The steady state $\frac{d^2 V}{dx^2} = 0 \longrightarrow V(x) = Ax + B$

$V(0) = T_0 \longrightarrow B = T_0$

$V(L) = T_1 \longrightarrow A = \frac{T_1 - T_0}{L}$

$$V(x) = T_0 + \frac{(T_1 - T_0)}{L} x$$

The transient solution

where $W(x,t) = u(x,t) - V(x)$ and $W(x,t)$ satisfies

$$\frac{\partial^2 W}{\partial x^2} = \frac{1}{k}\frac{\partial W}{\partial t} \qquad 0 < x < L \qquad t > 0$$

$W(0,t) = 0$, $W(L,t) = 0$ and

$$W(x,0) = \frac{T_1 - T_0}{L} x + T_0$$

According to our calculations above, what has the form

$$W(x,t) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x \exp -\lambda_n^2 kt$$

and the initial conditions gives

$$W(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = g(x) \qquad 0<x<L$$

The coefficient b_n is given by

$$b_n = \frac{2}{L}\int_0^L \left[-T_0 + (T_0 - T_1)\frac{x}{L}\right] \sin \frac{n\pi x}{L}\, dx = \frac{2}{n\pi}\left[T_0 + (-1)^n T_1\right]$$

and then the complete solution is

$$u(x,t) = W(x,t) + T_0 + (T_1 - T_0)\frac{x}{L}$$

where

$$W(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{T_0 + (-1)^n T_1}{n}\right] \sin \frac{n\pi x}{L} \exp -\frac{n^2\pi^2}{L^2} kt$$

Example III Insulated Bar

Let us suppose that the ends of the bar at $x=0$ and $x=L$ are insulated instead of being held at constant temperatures. The boundary value initial value problem which describes the temperature in this rod is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad 0<x<L \quad t>0$$

$$\frac{\partial u(0,t)}{\partial x} = 0 \qquad \frac{\partial u(L,t)}{\partial x} = 0 \qquad t>0$$

$$u(x,0) = f(x) \qquad 0<x<L,$$

where $f(x)$ is supposed to be a given function. Since the boundary conditions and the partial differential equation are homogeneous in this example. It is not necessary for us to establish the transient problem.

We may look for $u(x,t)$ directly. Let

$u(x,t) = \phi(x)T(t)$,
the heat equation becomes

$$\phi''(x)T(t) = \frac{1}{k}\phi(x)T'(t)$$

By separating the variables

$$\phi''/\phi = \frac{T'}{kT} = -\lambda^2 \qquad 0<x<L$$

$$\phi'' + \lambda^2\phi(x) = 0 \qquad 0<x<L \qquad T'(t) + \lambda^2 k T(t) = 0 \quad t>0$$

The boundary conditions on u can be translated on $\phi(x)$, because they are homogeneous conditions

$$\frac{\partial u(0,t)}{\partial x} = \phi'(0)T(t) = 0 \Rightarrow \phi'(0) = 0$$

$$\frac{\partial u(L,t)}{\partial x} = \phi'(L)T(t) = 0 \Rightarrow \phi'(L) = 0$$

We now have a homogeneous differential equation for $\phi(x)$ together with homogeneous boundary conditions

$$\phi''(x) + \lambda^2\phi(x) = 0 \qquad 0<x<L$$

$$\phi'(0) = 0 \quad \text{and} \quad \phi'(L) = 0$$

A problem of this kind is called an eigenvalue problem. We are looking for those values of the parameter λ^2 for which nonzero solutions exist. Those values are called eigenvalues, and the corresponding solutions are called eigenfunctions. The general solutions of the differential equation is

$$\phi(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

$$\phi'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$$

using the boundary conditions

$$\phi'(0) = 0 \qquad B = 0$$

$$\phi'(L) = 0 \qquad -A\lambda\sin\lambda L = 0$$

$\lambda_n = \frac{n\pi}{L}$ $\quad n = 0, 1, 2, \ldots$ and

$\phi_n(x) = \cos\lambda_n x \quad T_n = \exp -\lambda_n^2 kt$

$u_n(x,t) = \cos\lambda_n x \; \exp -\lambda_n^2 kt$

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\lambda_n x \; \exp -\lambda_n^2 kt$$

$$u(x,0) = a_0 + \sum_{n=1} a_n \cos\lambda_n x = f(x)$$

where.

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

or

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Example 4

Different boundary conditions:

In many cases, boundary conditions at the two end-points could be with different kinds. We shall solve the problem of finding the temperature in a rod having one insulated end and the other is kept at constant temperature.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad 0<x<L \qquad t>0$$

$u(0,t) = T_0 \qquad t>0$

$\frac{\partial u(L,t)}{\partial x} = 0 \qquad t>0 \quad$ and $u(x,0) = f(x)$

Soln

a) A steady state solution

Let $\quad u(x,t) = w(x,t) + v(x)$

where $\quad v''(x) = 0$ leads to $\quad v(x) = Ax + B$

$V(0) = T_0 \quad \Rightarrow \quad B = T_0$
$V'(L) = 0 \quad \Rightarrow \quad A = 0 \qquad$ Then $V(x) = T_0$

The transient solution $W(x,t)$ is satisfying the problem

$$\frac{\partial^2 W}{\partial x^2} = \frac{1}{k}\frac{\partial W}{\partial t} \qquad 0 < x < L \qquad t > 0$$

$$W(0,t) = 0 \qquad t > 0 \qquad \text{and} \qquad \frac{\partial W(L,t)}{\partial x} = 0 \qquad t > 0$$

$$W(x,0) = f(x) - T_0 = g(x) \qquad 0 < x < L$$

Using the product method to solve the tranisent problem ; $\quad W(x,t) = \phi(x)\,T(t)$

we get

$$\left.\begin{array}{l} \phi''(x) + \lambda^2 \phi = 0 \quad , \quad 0 < x < L \\ T' + \lambda^2 k T = 0 \end{array}\right\} \text{ with } \phi(0) = 0 \quad \phi'(L) = 0$$

The general solution of the first equation is

$$\phi(x) = a\cos\lambda x + b\sin\lambda x$$
$$\phi(0) = 0 \Rightarrow a = 0$$
$$\phi'(L) = 0 \rightarrow b\lambda\cos\lambda L = 0$$
$$\lambda_n = \frac{2n-1}{2L} \qquad n = 1, 2, \ldots$$

The eigenfunctions are given by the formula $\phi_n(x) = \sin\lambda_n x$ and the solution of the second equation is $T_n = \exp - \lambda_n^2 k t$.

The general solution of the transient problem is:

$$W(x,t) = \sum b_n \sin\lambda_n x \exp - \lambda_n^2 k t \quad \text{and}$$
$$W(x,0) = \sum b_n \sin\lambda_n x = g(x) \quad \text{with}$$
$$b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{2n-1}{2L}\pi x\right)dx$$

The final solution of the original problem is

$$u(x,t) = T_0 + \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp - \lambda_n^2 K t .$$

Example 5

Convection

The physical model is conduction of heat in a rod with insulated lateral surface whose left end is hold at a constant temperature and the right end is exposed to convective heat transfer

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K} \frac{\partial u}{\partial t} \qquad 0 < x < L \qquad t > 0$$

$$u(0,t) = T_0 \qquad t > 0$$

$$-k \frac{\partial u(L,t)}{\partial x} = h\left[u(L,t) - T_1\right] \qquad t > 0$$

$$u(x,0) = f(x) \qquad 0 < x < L$$

Sol$\underline{n}$

a – The steady state solution of this problem is

$V(x) = Ax + B$

$V(0) = B = T_0 \qquad V'(x) = A$

$$-kA = h\left[AL + T_0 - T_1\right]$$

$$A = \frac{h[T_1 - T_0]}{k + hL}$$

$$V(x) = T_0 + \frac{xh(T_1 - T_0)}{k + hL}$$

b. The transient solution is

$$W(x,t) = u(x,t) - V(x)$$

and by direct substitution, we find that

$$\frac{\partial^2 W}{\partial x^2} = \frac{1}{k}\frac{\partial W}{\partial t} \qquad 0<x<L \qquad t>0$$

$$W(0,t) = 0 \qquad t>0$$

$$hW(L,t) + k\frac{\partial W}{\partial x}(L,t) = 0 \qquad t>0$$

$$W(x,0) = f(x) - V(x) = g(x) \qquad 0<x<L$$

c- Let $W(x,t) = \phi(x)\,T(t)$

$$\phi'' + \lambda^2\phi = 0 \qquad 0<x<L \qquad (1)$$
$$T' + \lambda^2 k T = 0 \qquad t>0$$

with the boundary conditions

$$\phi(0) = 0 \qquad k\phi'(0) + h\phi(L) = 0$$

The general solution of the differential equation (1) is

$$\phi(x) = a\cos\lambda x + b\sin\lambda x$$

The boundary condition at zero requires that $\phi(0)=0$ leading to $\phi(x) = b\sin\lambda x$

Now at the other boundary

$$k\phi'(L) + h\phi(L) = 0$$
$$b[k\lambda\cos\lambda L + h\sin\lambda L] = 0$$

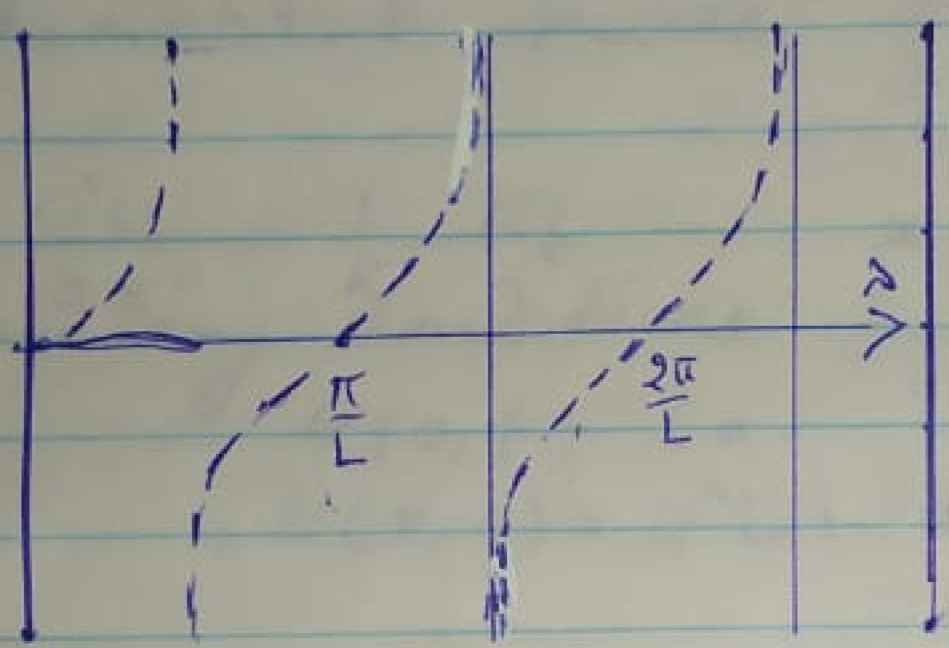
Discasding the possibilities $b=0$ and $\lambda=0$ which both lead to the trivial solution, we are left with the equation

$$k\lambda\cos\lambda L + h\sin\lambda L = 0$$
$$\text{or} \quad \tan\lambda L = -\frac{k\lambda}{h}$$

This Equation is called transcendental equation, which can not be solved exactly. However from sketches of the graphs of $\tan(\lambda L)$ and $-\frac{k\lambda}{h}$, we see that there are an infinite number of solutions

λ_n, $n = 1, \dots \infty$ and that for very large n, λ_n is given approximately by

$$\lambda_n = \frac{(2n-1)}{2}\frac{\pi}{L}$$



Thus we have for each $n = 1, 2, \dots$ an eigenvalue λ_n^2 and an eigenfuction Φ_n. Accompanying $\Phi_n(x)$ is the function

$$T_n(t) = \exp - \lambda_n^2 t k$$

Therefore, the transient solution will have the form

$$W(x,t) = \sum_{n=1} b_n \sin \lambda_n x \exp - \lambda_n^2 k t \qquad 0 < x < L \text{ and}$$

$$b_n = \int_0^L g_L(x) \sin \lambda_n x \, dx \Big/ \int_0^L \sin^2 \lambda_n x \, dx$$

Using this formula, b_n can be calculated and inserted into the formula for $W(x,t)$. Then we may put together the solution $u(x,t)$ of the original problem

$$u(x,t) = V(x) + W(x,t)$$

$$= T_0 + \frac{xh(T_1 - T_0)}{(K + hL)} + \sum_{n=1} b_n \sin \lambda_n x \exp - \lambda_n^2 k t$$

Example 6

Semi infinite bar

If the bar is very long and one end is not known to us, such case is treated as a semi infinite bar. The mathematical model of this problem is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} u_t \qquad x>0,\ t>0$$

with boundary conditions

$$u(0,t) = 0 \qquad t>0 \quad \text{and}$$
$$u(x,0) = f(x) \qquad x>0$$

Soln.

We shall use the separation of variable technique. This technique is based on the assumption of the solution to be

$$u(x,t) = \phi(x)\,T(t)$$

Thus we have

$$\phi''(x) + \lambda^2 \phi(x) = 0$$
$$T'(t) + \lambda^2 k T(t) = 0$$

There is just one boundary condition on u, which requires that $\phi(0) = 0$

The boundedness condition also requires that $\phi(x)$ remains finite as $x \to \infty$. The solution of the first ordinary differential equation is

$$\phi(x) = A\cos\lambda x + B\sin\lambda x$$

$$\psi(0) = 0 \qquad A = 0 \qquad \phi(x) = B\sin\lambda x$$

The solution of the second ordinary differential equation is $T(t) = \exp -\lambda^2 k t$

For any value of λ^2, we have

$$u(x,t;\lambda) = \sin\lambda x\, e^{-\lambda^2 k t}$$

which satisfies the partial differential equation, the boundary condition and the boundedness condition. The partial differential equation and the boundary condition are homogeneous, therefore any linear combination of solution is a solution. Since the parameter λ may take on any value the appropriate linear combination is an integral. So u should have the form

$$u(x,t) = \int_0^\infty B(\lambda) \sin \lambda x \exp -\lambda^2 \kappa t \, d\lambda$$

The initial condition will be satisfied if $B(\lambda)$ is chosen to make

$$u(x,0) = \int_0^\infty B(\lambda) \sin \lambda x \, dx = f(x)$$

$$B(\lambda) = \frac{2}{\pi} \int_0^a f(x) \sin \lambda x \, dx$$

Example (7)

Infinite Rod

If we wish to study heat conduction in the center of a very long rod, we may assume that it extends from $-\infty$ to ∞, then there are no boundary conditions and the problem to be solved is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa} \frac{\partial u}{\partial t} \qquad -\infty < x < \infty \qquad t > 0$$

$u(x,0) = f(x) \qquad -\infty < x < \infty$

$|u(x,t)|$ bounded as $x \Rightarrow \mp \infty$

Employing the same technique as before

$$U(x,t) = \Phi(x) T(t)$$

$$T'(t) + \lambda^2 \kappa T = 0 \qquad t > 0$$

$$\Phi''(x) + \lambda^2 \Phi(x) = 0 \qquad -\infty < x < \infty$$

The signs have been chosen to make $\phi(x)$ satisfying the boundness conditions

$$\phi(x) = A\cos(\lambda x) + B\sin(\lambda x)$$
$$\phi(t) = \exp - \lambda^2 k t$$

We combine the solution $\phi(x)T(t)$ in the form of an integral to obtain

$$u(x,t) = \int_0^\infty \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right] e^{-\lambda^2 k t}\, d\lambda \quad (1)$$

The initial condition is satisfied by choosing

$$\begin{matrix} A(\lambda) \\ B(\lambda) \end{matrix} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \left[\begin{matrix}\cos\lambda x \\ \sin\lambda x\end{matrix}\right. dx \quad (2)$$

for, when $t=0$

the initial condition has the form of a Fourier integral

$$f(x) = \int_0^\infty \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right] d\lambda$$

In this case, we can derive some very interesting results. Changing the variable of integration in eqn (2) to ξ and substituting the formula for $A(\lambda)$ and $B(\lambda)$ into equation (1), we get

$$u(x,t) = \frac{1}{\pi}\int_0 \left[\int_{-\infty}^{\infty} f(\xi)\left[\cos\lambda(x-\xi)\right] d\xi\; e^{-k\lambda^2 t}\, d\lambda\right.$$

Reversing the order of integration, we get

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\int_0^\infty \cos\lambda(x-\xi)\; e^{-k\lambda^2 t}\, d\lambda\, d\xi$$

The inner integral can be computed by complex method of integration. It is known to equal

$$\sqrt{\frac{\pi}{4kt}}\, e^{-\frac{(x-\xi)^2}{4kt}} \qquad t>0$$

This gives us, finally, a new form for the temperature distribution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(\xi) \exp -\frac{(\xi-x)^2}{4kt}\, d\xi \quad (3)$$

Each of these solutions (1) and (3) has its own advantages. For simple problems we may able to evaluate the coefficients $A(\lambda)$ and $B(\lambda)$ in equation (2). But it is a rare case indeed when the integral in equation (1) can be evaluated analytically. The same is true for the integral in equation (3). Thus if the value of u at a specific x and t is needed, either integral would be calculated analytically. For large value of kt, the exponential factor in the integrand of equation (1) will be nearly zero, except for small λ. Thus equation (1) is approximately

$$u(x,t) = \int_0^R [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\, e^{-\lambda^2 kt}\, d\lambda$$

for R not large and the right-hand side may be found to high accuracy with little effort. On the other hand, if kt is small, the exponential in the integrand of equation (3) will be nearly zero, except for ξ near x. The approximation

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{x-h}^{x+h} f(\xi) \exp -\frac{(x-\xi)^2}{4kt}\, d\xi$$

is satisfactory for h not large and again numerical techniques are easily applied to the right hand side.
The expression in equation (3) has a number of dis-advantages also. It requires no intermediate integ-rations (compared with equation (2)). It shows dire-ctly the influence of initial conditions on the solution. Moreover, the function $f(x)$ need not to satisfy the restriction

$$\int_{-\infty}^{\infty} f(x)\,dx < \infty$$

in order for equation (3) to satisfy the original problem.

Summary of the solution of the Heat Eqn.

We can outline the method we have been using to solve linear boundary value - initial value problems. Up to this moment we have seen only homogeneous partial differential equation, but a non-homogeneity which is independent of t can be treated by the same technique.

1) If the partial differential equation or a boundary condition or both are not homogeneous, first find the function $V(x)$, independent of t, which satisfies the partial differential equation and the boundary conditions. Since $V(x)$ does not depend on t, the partial differential equation applied to $V(x)$ becomes an ordinary differential equation. Finding $V(x)$ is just a matter of solving a two-point boundary value problem.

2) Determine the initial value problem satisfied by

the transient solution.

$$W(x,t) = u(x,t) - V(x)$$

This must be homogeneous problem. That is the partial differential equation and the boundary conditions (but not usually the initial condition) are satisfied by the constant function 0.

3) Assuming that $W(x,t) = \phi(x)T(t)$, separate the partial differential equation into ordinary differential equations one for $\phi(x)$ and one for $T(t)$ linked by the separation constant $-\lambda^2$. Also reduces the boundary conditions to conditions on $\phi(x)$ alone.

4) Solve the eigenvalue problem for $\phi(x)$, that is find the values of λ^2 for which the eigenvalue problem has nonzero solutions. Label the eigenfunctions and the eigenvalues by $\phi_n(x)$ and λ_n^2 respectively

5) Solve the ordinary differential equation for the time factors; $T_n(t)$.

6) Form the general solution of the homogeneous problem a sum of constant multiplied by the product

$$W(x,t) = \sum_{n=1}^{\infty} C_n \phi_n(x) T_n(t)$$

7) Choose the C_n so that the initial condition is satisfied. This may or may not be a routine Fourier series problem

8) Form the solution of the original problem

$$u(x,t) = V(x) + W(x,t)$$

and show whether the conditions are satisfied or not.

Problems

Find the steady-state solution, the associated eigenvalue problem, and the complete solution for each of the conduction problems below.

1- $u_{xx} = \frac{1}{k} u_t \qquad 0 < x < a \qquad t > 0$

$u(0,t) = T_0 \qquad u(a,t) = T_0 \qquad t > 0$

$u(x,0) = T_1 \qquad 0 < x < a$

2- $u_{xx} = \gamma^2 u + \frac{1}{k} u_t \qquad 0 < x < a \qquad t > 0$

$u(0,t) = T_0 \qquad u(a,t) = T_0 \qquad t > 0$

$u(x,0) = T_1 \qquad 0 < x < a$

3. $u_{xx} - r = \frac{1}{k} u_t \qquad 0 < x < a \qquad t > 0$

$u(0,t) = T_0 \qquad u(a,t) = T_0 \qquad t > 0$

$u(x,t) = T_1 \qquad 0 < x < a$

where r is constant.

$u_{xx} = \frac{1}{k} u_t \qquad 0 < x < a \qquad t > 0$

$u(0,t) = T_0 \qquad \frac{\partial u}{\partial x}(a,t) = 0 \qquad t > 0$

$u(x,0) = T_1 \frac{x}{a} \qquad 0 < x < a$

$u_{xx} - \gamma^2 u = \frac{1}{k} u_t \qquad 0 < x < a \qquad t > 0$

$\frac{\partial u(0,t)}{\partial x} = 0 \qquad \frac{\partial u}{\partial x}(a,t) = 0 \qquad t > 0$

$u(x,0) = T_1 \frac{x}{a} \qquad 0 < x < a$

$u_{xx} = \frac{1}{k} u_t \qquad 0 < x < a \qquad t > 0$

$u(0,t) = 0 \qquad u(a,t) = T_0 \qquad t > 0$

$u(x,0) = 0 \qquad 0 < x < a$

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<a \qquad t>0$$
$$u(0,t) = T_0 \qquad u(a,t) = T_0 \qquad t>0$$
$$u(x,0) = T_0 \qquad 0<x<a$$

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<a \qquad t>0$$
$$\frac{\partial u(0,t)}{\partial x} = \frac{\Delta T}{a} \qquad \frac{\partial u(a,t)}{\partial x} = \frac{\Delta T}{a} \qquad t>0$$
$$u(x,0) = T_0 \qquad 0<x<a$$

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<a \qquad t>0$$
$$u(0,t) = T_0 \qquad \frac{\partial u(a,t)}{\partial x} = 0 \qquad t>0$$
$$u(x,0) = T_1 \qquad 0<x<a$$

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<a \qquad t>0$$

$$\frac{\partial u(0,t)}{\partial x} = 0 \qquad \frac{\partial u(a,t)}{\partial x} = 0 \qquad t>0$$
$$u(x,0) = \begin{cases} T_0 & 0<x<a/2 \\ T_1 & a/2<x<a \end{cases}$$

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<\infty \qquad t>0$$
$$u(0,t) = T_0 \qquad t>0$$
$$u(x,0) = T_0(1 - \exp{-\alpha x}) \qquad x>0$$

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<\infty \qquad t>0$$

$$u(0,t) = 0 \qquad u(x,0) = \begin{cases} T_0 & 0<x<a \\ 0 & x>a \end{cases}$$

$$u_{xx} = \frac{1}{k} u_t \qquad 0 < x < \infty \qquad t > 0$$

$$\frac{\partial u(0,t)}{\partial x} = 0 \quad t > 0 \qquad u(x,0) = \begin{cases} T_0 & 0 < x < a \\ 0 & x > a \end{cases}$$

$$u_{xx} = \frac{1}{k} u_t \qquad -\infty < x < \infty \qquad t > 0$$
$$u(x,0) = \exp -\alpha |x| \qquad -\infty < x < \infty$$

$$u_{xx} = \frac{1}{k} u_t \qquad -\infty < x < \infty \qquad t > 0$$

$$u(x,0) = \begin{cases} 0 & -\infty < x < 0 \\ T_0 & 0 < x < a \\ 0 & a < x < \infty \end{cases}$$

solve the eigenvalue problem by setting $\phi(\rho) = \psi(\rho)/\rho$

$$\frac{1}{\rho^2}(\rho^2 \phi')' + \lambda^2 \phi = 0 \qquad 0 < \rho < a$$

$\phi(0)$ is bounded and $\phi(a) = 0$
Is this a regular sturm-Liouville problem?
Are the eigenfunctions Orthogonal,

solve the problem for heat conduction in a sphere
let $u(\rho,t) = v(\rho,t)/\rho)$

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) = \frac{1}{k}\frac{\partial u}{\partial t} \qquad 0 < \rho < a \qquad t > 0$$

$u(0,t)$ is bounded $\qquad u(a,t) = 0 \qquad t > 0$
$u(\rho,t) = T_0 \qquad 0 < \rho < a$

state and solve the eigenvalue problem associated with

$$e^{-x}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial u}{\partial x}\right)=\frac{1}{k}\frac{\partial u}{\partial t} \qquad 0<x<a \quad t>0$$

$u(0,t)=0$ $\qquad \frac{\partial u(a,t)}{\partial x}=0$

19 Find the steady state solution of

$$\frac{\partial u}{\partial x^2}+\gamma^2(u-T)=\frac{1}{k}u_t \qquad 0<x<a \quad t>0$$

$u(0,t)=T_0$, $\frac{\partial u}{\partial x}(a,t)=0 \quad t>0$

where $T(x)=T_0+\frac{\Delta T}{a}x \qquad 0<x<a$

20 suppose that $u(x,t)$ is a positive function that satisfies

$$u_{xx}=\frac{1}{k}u_t$$

show that the function $W(x,t)=-\frac{2}{u}\frac{\partial u}{\partial x}$ satisfies the nonlinear partial differential equation

$$\frac{\partial W}{\partial t}+W\frac{\partial W}{\partial x}-\frac{\partial^2 W}{\partial x^2}=0 \quad \text{(Burger's eqn)}$$

21 Find a solution of the Burger's equation that satisfies the boundary conditions

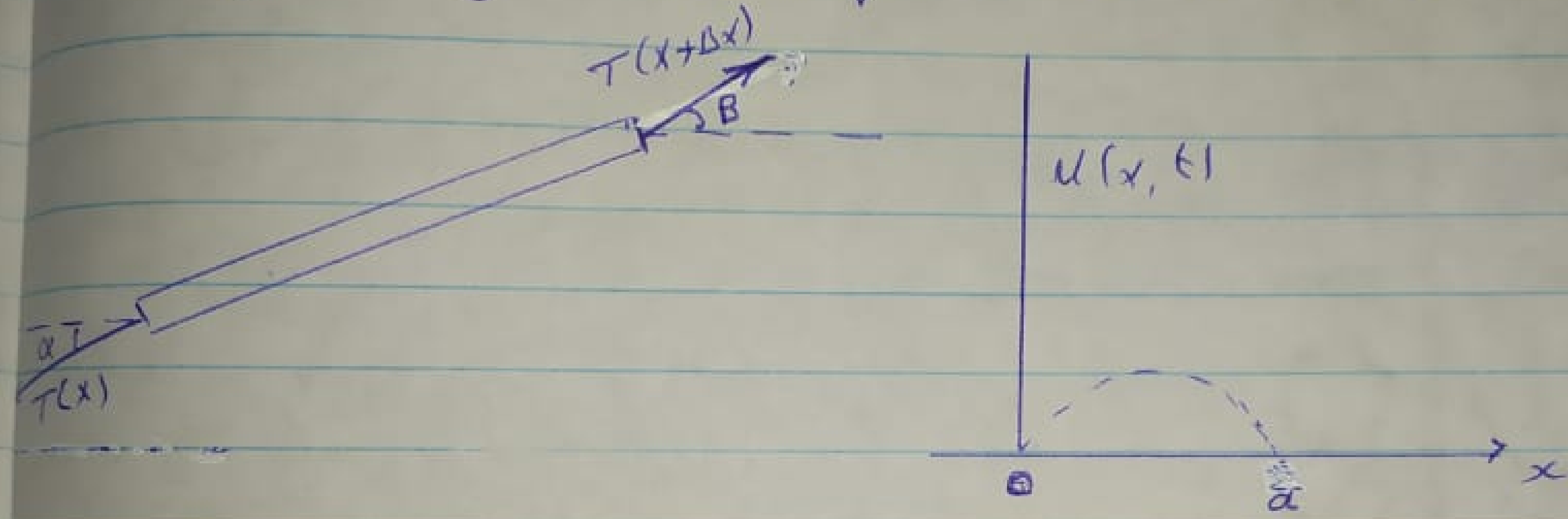
$W(0,t)=0 \qquad W(1,t)=0 \qquad t>0$

$W(x,0)=1 \qquad 0<x<1$

The Wave Equation

The Vibrating String:

For a simple example of wave equation, we consider the motion of a string which is fixed at its ends



$u(x,t)$ is the displacement of the string (unknown).

We assume that

i) the string is perfectly flexible, offers no resistance to bending. This means that the forces exerted on the element are tangent to its mid-line at the point where they act.

ii) A point on the string moves in the vertical direction

Applying Newton's second law on the element at x,

$T(x+\Delta x)\cos\beta - T(x)\cos(\alpha) = 0$, then

$T(x+\Delta x)\cos\beta = T\cos(\alpha) = T$

where T is a constant

$$T(x+\Delta x) = \frac{T}{\cos\beta}$$

$$T(x) = \frac{T}{\cos(\alpha)}$$

Newton's law in the vertical direction yields

$$T(x+\Delta x)\sin\beta - T(x)\sin\alpha - mg = m\frac{\partial^2}{\partial t^2}u(x,t)$$

Hence, the above equation can be written in the form