19. Find the Laurent series expansion of the function $\dfrac{(z^2-6z-1)}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$. **(JNTU 2001, 2005 May, 2006 Nov., 2008 Nov.)**

Solution:

$$\frac{(z^2-6z-1)}{(z-1)(z-3)(z+2)} = \frac{A}{(z-1)} + \frac{B}{(z-3)} + \frac{C}{(z+2)}$$

$$A = \lim_{z\to 1} \frac{z^2-6z-1}{(z-3)(z+2)} = 1$$

$$B = \lim_{z\to 3} \frac{z^2-6z-1}{(z-1)(z+2)} = -1$$

$$C = \lim_{z\to -2} \frac{z^2-6z-1}{(z-1)(z-3)} = 1$$

$$\frac{3}{|z+2|} < 1, \quad \frac{|z+2|}{5} < 1$$

$$\therefore \quad f(z) = \frac{1}{(z-1)} - \frac{1}{(z-3)} + \frac{1}{(z+2)}$$

$$= \frac{1}{(z+2-3)} - \frac{1}{(z+2-5)} + \frac{1}{(z+2)}$$

$$= \frac{1}{(z+2)\left(1-\frac{3}{z+2}\right)} + \frac{1}{5\left(1-\frac{z+2}{5}\right)} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\left(1-\frac{3}{z+2}\right)^{-1} + \frac{1}{5}\left(1-\frac{z+2}{5}\right)^{-1} + \frac{1}{z+2}$$

$$= \frac{1}{(z+2)}\left(1+\frac{3}{(z+2)}+\frac{3^2}{(z+2)^2}-\frac{3^3}{(z+2)^3}+\cdots\right)$$

$$+\frac{1}{5}\left(1+\frac{z+2}{5}+\frac{(z+2)^2}{5^2}+\frac{(z+2)^3}{5^3}+\cdots\right)+\frac{1}{z+2}$$

$$= \sum_{n=0}^{\infty}\frac{3^n}{(z+2)^{n+1}} + \sum_{n=0}^{\infty}\frac{(z+2)^n}{5^{n+1}} + \frac{1}{z+2}$$

$$= \sum_{n=0}^{\infty}\left[\frac{3^n}{(z+2)^{n+1}} + \frac{(z+2)^n}{5^{n+1}}\right] + \frac{1}{z+2}.$$

24. Find the Laurent series for $f(z) = \dfrac{z}{(z^2-1)(z^2+4)}$ if (i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$.

Solution:
$$\frac{z}{(z^2-1)(z^2+4)} = \frac{Az}{(z^2-1)} + \frac{Bz}{(z^2+4)}$$

On solving, we get

$$A = \frac{1}{5},\ B = -\frac{1}{5}$$

$$f(z) = \frac{z}{5(z^2-1)} - \frac{z}{5(z^2+4)}$$

(i) $$-\frac{z}{5}(1-z^2)^{-1} - \frac{z}{20}\left(1+\frac{z^2}{4}\right)^{-1} = -\frac{z}{5}(1+z^2+z^4+\cdots)$$

$$-\frac{z}{20}\left(1-\frac{z^2}{4}+\frac{z^4}{4^2}+\cdots\right)$$

$$= -\frac{1}{20}\left[\sum_{n=1}^{\infty} 4z^{2n-1} + \sum_{n=1}^{\infty} \frac{z^{2n-1}(-1)^{n-1}}{4^{n-1}}\right]$$

(ii) $$1 < |z| < 2,\ \frac{1}{|z|} < 1, \frac{|z|}{2} < 1$$

$$\frac{z}{5(z^2-1)} - \frac{z}{5(z^2+4)} = \frac{z}{5z^2\left(1-\frac{1}{z^2}\right)} - \frac{z}{20\left(1+\frac{z^2}{4}\right)}$$

$$= \frac{1}{5z}\left(1-\frac{1}{z^2}\right)^{-1} - \frac{z}{20}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$= \frac{1}{5z}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\cdots\right) - \frac{z}{20}\left(1-\frac{z^2}{4}+\frac{z^4}{16}-\cdots\right)$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{z^{2n-1}} - \frac{1}{20}\sum_{n=1}^{\infty}\frac{z^{2n-1}}{4^{n-1}}$$

(iii) $|z| > 2,\ 2 < |z|,\ \dfrac{2}{|z|} < 1,\ \dfrac{1}{|z|} < 1$

$$f(z) = \frac{z}{5(z^2-1)} - \frac{z}{5(z^2+4)}$$

$$= \frac{1}{5z\left(1-\frac{1}{z^2}\right)} - \frac{1}{5z\left(1+\frac{4}{z^2}\right)}$$

$$= \frac{1}{5z}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right) - \frac{1}{5z}\left(1 - \frac{4}{z^2} + \frac{4^2}{z^4} - \frac{4^3}{z^6} + \cdots\right)$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^{2n-1}} - \frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4^{n-1}}{z^{2n-1}}$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^{2n-1}}[1-4^{n-1}].$$

30. Expand $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ in powers of z

(i) with in the unit circle about the origin

(ii) within the annular region between the concentric circle about the origin having radii 1 and 2, respectively

(iii) the exterior to the circle with centre as origin and radius 2.

Solution:

(i)
$$\frac{z+3}{z(z^2-z-2)} = \frac{A}{z} + \frac{B}{(z-2)} + \frac{C}{(z+1)}$$

$$A = \lim_{z\to 0}\frac{(z+3)}{(z^2-z-2)} = \frac{-3}{2}$$

$$B = \lim_{z\to 2}\frac{(z+3)}{z(z+1)} = \frac{5}{6}$$

$$C = \lim_{z\to -1}\frac{(z+3)}{z(z-2)} = \frac{2}{3}$$

$$\therefore \quad \frac{z+3}{z(z^2-z-2)} = -\frac{3}{2z} + \frac{5}{6(z-2)} + \frac{2}{3(z+1)}$$

$$= -\frac{3}{2z} + \frac{5}{12\left(1-\frac{z}{2}\right)} + \frac{2}{3(z+1)}$$

$$= -\frac{3}{2z} - \frac{5}{12}\left(1-\frac{z}{2}\right)^{-1} + \frac{2}{3}(1+z)^{-1}$$

$$= -\frac{3}{2z} - \frac{5}{12}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$

$$+ \frac{2}{3}(1 - z + z^2 - z^3 + \cdots)$$

$$= -\frac{3}{2z} + \sum_{n=0}^{\infty}\left[\frac{2(1)^n z^n}{3} - \frac{5}{12}\left(\frac{z}{2}\right)^n\right]$$

(ii) $1 < |z| < 2,\quad \frac{|z|}{2} < 1,\quad \frac{1}{|z|} < 1$

$$\therefore \qquad f(z) = -\frac{3}{2z} - \frac{5}{12}\left(1 - \frac{z}{2}\right)^{-1} + \frac{2}{3z}\left(1 + \frac{1}{z}\right)^{-1}$$

$$= -\frac{3}{2z} - \frac{5}{12}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots\right) + \frac{2}{3z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$= -\frac{3}{2z} + \sum_{n=0}^{\infty}\left[\frac{2}{3z}\frac{(-1)^n}{z^n} - \frac{5}{12}\left(\frac{z}{2}\right)^n\right]$$

(iii) $|z| > 2,\quad 2 < |z|,\quad \frac{2}{|z|} < 1$

$$\therefore \qquad \frac{z+3}{z(z^2 - z - 2)} = -\frac{3}{2z} + \frac{5}{6z}\left(1 - \frac{2}{z}\right)^{-1} + \frac{2}{3z}\left(1 + \frac{2}{z}\right)^{-1}$$

$$= -\frac{3}{2z} + \sum_{n=0}^{\infty}\left[\frac{5}{6z}\left(\frac{2}{z}\right)^n + \frac{2(-1)^n}{3z}\left(\frac{2}{z}\right)^n\right].$$

5. THE RESIDUE THEOREM

Let z_0 be an isolated singular point of $f(z)$. We are going to find the value of $\oint_C f(z)\,dz$ around a simple closed curve C surrounding z_0 but inclosing no other singularities. Let $f(z)$ be expanded in the Laurent series (4.1) about $z = z_0$ that converges near $z = z_0$. By Cauchy's theorem (V), the integral of the "a" series is zero since this part is analytic. To evaluate the integrals of the terms in the "b" series in (4.1), we replace the integrals around C by integrals around a circle C' with center at z_0 and radius ρ as in (3.6), (3.7), and Figure 3.1. Along C', $z = z_0 + \rho e^{i\theta}$; calculating the integral of the b_1 term in (4.1), we find

$$\oint_C \frac{b_1\,dz}{(z - z_0)} = b_1 \int_0^{2\pi} \frac{\rho i e^{i\theta}\,d\theta}{\rho e^{i\theta}} = 2\pi i b_1. \tag{5.1}$$

It is straightforward to show (Problem 1) that the integrals of all the other b_n terms are zero. Then $\oint_C f(z)\,dz = 2\pi i b_1$, or since b_1 is called the residue of $f(z)$ at $z = z_0$, we can say

$$\oint_C f(z)\,dz = 2\pi i \cdot \text{residue of } f(z) \text{ at the singular point inside } C.$$

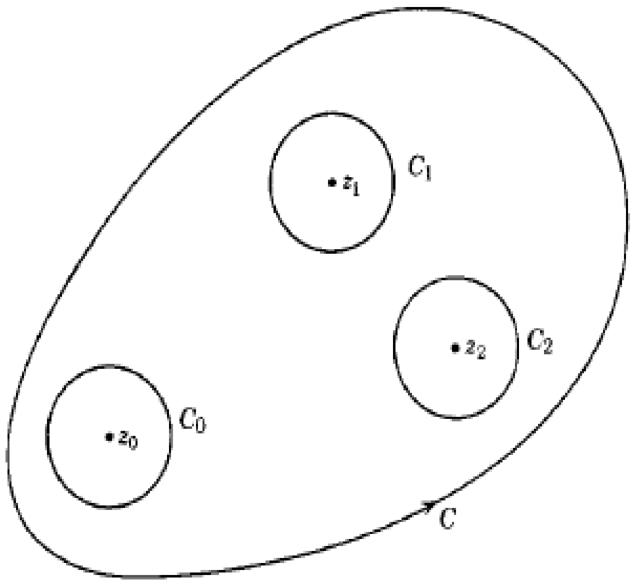

FIGURE 5.1

$$\oint_C f(z)\,dz = 2\pi i \cdot \text{sum of the residues of } f(z) \text{ inside } C, \tag{5.2}$$

where the integral around C is in the counterclockwise direction.

6. METHODS OF FINDING RESIDUES

A. Laurent Series If it is easy to write down the Laurent series for $f(z)$ about $z = z_0$ that is valid near z_0, then the residue is just the coefficient b_1 of the term $1/(z - z_0)$. *Caution:* Be sure you have the expansion about $z = z_0$; the series you have memorized for e^z, sin z, etc., are expansions about $z = 0$ and so can be used only for finding residues at the origin (see Section 4, Example 3). Here is another example: Given $f(z) = e^z/(z-1)$, find the residue, $R(1)$, of $f(z)$ at $z = 1$. We want to expand e^z in powers of $z - 1$; we write

$$\frac{e^z}{z-1} = \frac{e \cdot e^{z-1}}{z-1} = \frac{e}{z-1}\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right]$$

$$= \frac{e}{z-1} + e + \cdots .$$

Then the residue is the coefficient of $1/(z-1)$, that is,

$$R(1) = e.$$

B. Simple Pole If $f(z)$ has a simple pole at $z = z_0$, we find the residue by multiplying $f(z)$ by $(z - z_0)$ and evaluating the result at $z = z_0$ (Problem 10).

Example 1. Find $R(-\frac{1}{2})$ and $R(5)$ for

$$f(z) = \frac{z}{(2z+1)(5-z)}.$$

Multiply $f(z)$ by $(z + \frac{1}{2})$ [*Caution:* not by $(2z + 1)$] and evaluate the result at $z = -\frac{1}{2}$. We find

$$(z + \tfrac{1}{2})f(z) = (z + \tfrac{1}{2})\frac{z}{(2z+1)(5-z)} = \frac{z}{2(5-z)},$$

$$R(-\tfrac{1}{2}) = \frac{-\frac{1}{2}}{2(5+\frac{1}{2})} = -\tfrac{1}{22}.$$

Similarly,

$$(z-5)f(z) = (z-5)\frac{z}{(2z+1)(5-z)} = -\frac{z}{2z+1},$$

$$R(5) = -\tfrac{5}{11}.$$

Example 2. Find $R(0)$ for $f(z) = (\cos z)/z$.

Since $zf(z) = \cos z$, we have

$$R(0) = (\cos z)_{z=0} = \cos 0 = 1.$$

(6.1) $R(z_0) = \lim_{z \to z_0} (z - z_0) f(z)$ when z_0 is a simple pole.

Example 3. Find the residue of $\cot z$ at $z = 0$.
By (6.1),

$$R(0) = \lim_{z \to 0} \frac{z \cos z}{\sin z} = \cos 0 \cdot \lim_{z \to 0} \frac{z}{\sin z} = 1 \cdot 1 = 1.$$

If, as often happens, $f(z)$ can be written as $g(z)/h(z)$, where $g(z)$ is analytic and not zero at z_0 and $h(z_0) = 0$, then (6.1) becomes

$$R(z_0) = \lim_{z \to z_0} \frac{(z - z_0)g(z)}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{z - z_0}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{1}{h'(z)} = \frac{g(z_0)}{h'(z_0)}$$

by L'Hôpital's rule or the definition of $h'(z)$ (Problem 11).
Thus we have

(6.2) $$R(z_0) = \frac{g(z_0)}{h'(z_0)} \quad \text{if} \quad \begin{cases} f(z) = g(z)/h(z), \text{ and} \\ g(z_0) = \text{finite const.} \neq 0, \text{ and} \\ h(z_0) = 0,\ h'(z_0) \neq 0. \end{cases}$$

Often (6.2) gives the most convenient way of finding the residue at a simple pole.

Example 4. Find the residue of $(\sin z)/(1 - z^4)$ at $z = i$.
By (6.2) we have

$$R(i) = \left.\frac{\sin z}{-4z^3}\right|_{z=i} = \frac{\sin i}{-4i^3} = \frac{e^{-1} - e}{(2i)(4i)} = \tfrac{1}{8}(e - e^{-1}) = \tfrac{1}{4} \sinh 1.$$

of the denominator.] Suppose $f(z)$ is written in the form $g(z)/h(z)$, where $g(z)$ and $h(z)$ are analytic. Then you can think of $g(z)$ and $h(z)$ as power series in $(z - z_0)$. If the denominator has the factor $(z - z_0)$ to *one* higher power than the numerator, then $f(z)$ has a simple pole at z_0. For example,

$$z \cot^2 z = \frac{z \cos^2 z}{\sin^2 z} = \frac{z(1 - z^2/2 + \cdots)^2}{(z - z^3/3! + \cdots)^2} = \frac{z(1 + \cdots)}{z^2(1 + \cdots)}$$

has a simple pole at $z = 0$. By the same method we can see whether a function has a pole of any order.

C. Multiple Poles When $f(z)$ has a pole of order n, we can use the following method of finding residues.

Multiply $f(z)$ by $(z - z_0)^m$, where m is an integer greater than or equal to the order n of the pole, differentiate the result $m - 1$ times, divide by $(m - 1)!$, and evaluate the resulting expression at $z = z_0$.

Example 5. Find the residue of $f(z) = (z \sin z)/(z-\pi)^3$ at $z = \pi$.

We take $m = 3$ to eliminate the denominator before differentiating; this is an allowed choice for m because the order of the pole of $f(z)$ at π is not greater than 3 since $z \sin z$ is finite at π. (The pole is actually of order 2, but we do not need this fact.) Then following the rule stated, we get

$$R(\pi) = \frac{1}{2!}\frac{d^2}{dz^2}(z \sin z)\bigg|_{z=\pi} = \tfrac{1}{2}[-z\sin z + 2\cos z]_{z=\pi} = -1.$$

RESIDUES AND THE RESIDUE THEOREM

13.23. If $f(z)$ is analytic everywhere inside and on a simple closed curve C except at $z = a$ which is a pole of order n so that

$$f(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

where $a_{-n} \neq 0$, prove that

(a) $\oint_C f(z)\,dz = 2\pi i a_{-1}$

(b) $a_{-1} = \lim_{z\to a} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\{(z-a)^n f(z)\}$.

(a) By integration, we have on using Problem 13.13

$$\oint_C f(z)\,dz = \oint_C \frac{a_{-n}}{(z-a)^n}\,dz + \cdots + \oint_C \frac{a_{-1}}{z-a}\,dz + \oint_C \{a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\}\,dz$$
$$= 2\pi i a_{-1}$$

Since only the term involving a_{-1} remains, we call a_{-1} the *residue* of $f(z)$ at the pole $z = a$.

(b) Multiplication by $(z-a)^n$ gives the Taylor series

$$(z-a)^n f(z) = a_{-n} + a_{-n+1}(z-a) + \cdots + a_{-1}(z-a)^{n-1} + \cdots$$

Taking the $(n-1)$st derivative of both sides and letting $z \to a$, we find

$$(n-1)!\,a_{-1} = \lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\{(z-a)^n f(z)\}$$

from which the required result follows.