13.24. Determine the residues of each function at the indicated poles.

(a) $\dfrac{z^2}{(z-2)(z^2+1)}$; $z = 2, i, -i$. These are simple poles. Then:

Residue at $z = 2$ is $\displaystyle\lim_{z\to 2}(z-2)\left\{\frac{z^2}{(z-2)(z^2+1)}\right\} = \frac{4}{5}.$

Residue at $z = i$ is $\displaystyle\lim_{z\to i}(z-i)\left\{\frac{z^2}{(z-2)(z-i)(z+i)}\right\} = \frac{i^2}{(i-2)(2i)} = \frac{1-2i}{10}.$

Residue at $z = -i$ is $\displaystyle\lim_{z\to -i}(z+i)\left\{\frac{z^2}{(z-2)(z-i)(z+i)}\right\} = \frac{i^2}{(-i-2)(-2i)} = \frac{1+2i}{10}.$

(b) $\dfrac{1}{z(z+2)^3}$; $z = 0, -2$. $z = 0$ is a simple pole, $z = -2$ is a pole of order 3. Then:

Residue at $z = 0$ is $\displaystyle\lim_{z\to 0} z\cdot\frac{1}{z(z+2)^3} = \frac{1}{8}.$

Residue at $z = -2$ is $\displaystyle\lim_{z\to -2}\frac{1}{2!}\frac{d^2}{dz^2}\left\{(z+2)^3\cdot\frac{1}{z(z+2)^3}\right\}$

$$= \lim_{z\to -2}\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{z}\right) = \lim_{z\to -2}\frac{1}{2}\left(\frac{2}{z^3}\right) = -\frac{1}{8}.$$

Note that these residues can also be obtained from the coefficients of $1/z$ and $1/(z+2)$ in the respective Laurent series [see Problem 13.22(e)].

(c) $\dfrac{ze^{zt}}{(z-3)^2}$; $z = 3$, a pole of order 2 or double pole. Then:

Residue is $\displaystyle\lim_{z\to 3}\frac{d}{dz}\left\{(z-3)^2\cdot\frac{ze^{zt}}{(z-3)^2}\right\} = \lim_{z\to 3}\frac{d}{dz}(ze^{zt}) = \lim_{z\to 3}(e^{zt}+zte^{zt})$

$$= e^{3t} + 3te^{3t}$$

(d) $\cot z$; $z = 5\pi$, a pole of order 1. Then:

Residue is $\displaystyle\lim_{z\to 5\pi}(z-5\pi)\cdot\frac{\cos z}{\sin z} = \left(\lim_{z\to 5\pi}\frac{z-5\pi}{\sin z}\right)\left(\lim_{z\to 5\pi}\cos z\right) = \left(\lim_{z\to 5\pi}\frac{1}{\cos z}\right)(-1)$

where we have used L'Hospital's rule, which can be shown applicable for functions of a complex variable.

13.26. Evaluate $\displaystyle\oint_C \frac{e^z\,dz}{(z-1)(z+3)^2}$ where C is given by (a) $|z| = 3/2$, (b) $|z| = 10$.

Residue at simple pole $z = 1$ is $\displaystyle\lim_{z\to 1}\left\{(z-1)\frac{e^z}{(z-1)(z+3)^2}\right\} = \frac{e}{16}$

Residue at double pole $z = -3$ is

$$\lim_{z\to -3}\frac{d}{dz}\left\{(z+3)^2\frac{e^z}{(z-1)(z+3)^2}\right\} = \lim_{z\to -3}\frac{(z-1)e^z - e^z}{(z-1)^2} = \frac{-5e^{-3}}{16}$$

(a) Since $|z| = 3/2$ encloses only the pole $z = 1$,

$$\text{the required integral} = 2\pi i\left(\frac{e}{16}\right) = \frac{\pi i e}{8}$$

(b) Since $|z| = 10$ encloses both poles $z = 1$ and $z = -3$,

$$\text{the required integral} = 2\pi i\left(\frac{e}{16} - \frac{5e^{-3}}{16}\right) = \frac{\pi i(e - 5e^{-3})}{8}$$

7.5. Find the residue of $F(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.

Solution

We have, as in Method 2 of Problem 7.4(b),

$$F(z) = \frac{\cos z \cosh z}{z^3 \sin z \sinh z} = \frac{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)}{z^3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)}$$

$$= \frac{\left(1 - \frac{z^4}{6} + \cdots\right)}{z^5\left(1 - \frac{z^4}{90} + \cdots\right)} = \frac{1}{z^5}\left(1 - \frac{7z^4}{45} + \cdots\right)$$

and so the residue (coefficient of $1/z$) is $-7/45$.

Another Method. The result can also be obtained by finding

$$\lim_{z\to 0} \frac{1}{4!} \frac{d^4}{dz^4}\left\{z^5 \frac{\cos z \cosh z}{z^3 \sin z \sinh z}\right\}$$

but this method is much more laborious than that given above.

7.6. Evaluate $\dfrac{1}{2\pi i} \oint\limits_C \dfrac{e^{zt}}{z^2(z^2 + 2z + 2)}\, dz$ around the circle C with equation $|z| = 3$.

Solution

The integrand $e^{zt}/\{z^2(z^2 + 2z + 2)\}$ has a double pole at $z = 0$ and two simple poles at $z = -1 \pm i$ [roots of $z^2 + 2z + 2 = 0$]. All these poles are inside C.

Residue at $z = 0$ is

$$\lim_{z\to 0} \frac{1}{1!} \frac{d}{dz}\left\{z^2 \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\right\} = \lim_{z\to 0} \frac{(z^2 + 2z + 2)(te^{zt}) - (e^{zt})(2z + 2)}{(z^2 + 2z + 2)^2} = \frac{t - 1}{2}$$

Residue at $z = -1 + i$ is

$$\lim_{z\to -1+i}\left\{[z - (-1 + i)]\frac{e^{zt}}{z^2(z^2 + 2z + 2)}\right\} = \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2}\right\} \lim_{z\to -1+i}\left\{\frac{z + 1 - i}{z^2 + 2z + 2}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1 + i)^2} \cdot \frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1 - i$ is

$$\lim_{z\to -1-i}\left\{[z - (-1 - i)]\frac{e^{zt}}{z^2(z^2 + 2z + 2)}\right\} = \frac{e^{(-1-i)t}}{4}$$

Then, by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i \text{ (sum of residues)} = 2\pi i\left\{\frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}\right\}$$

$$= 2\pi i\left\{\frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t\right\}$$

that is,

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t$$

13.27. If $|f(z)| \leqq \frac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 1$ and M are constants, prove that $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ where Γ is the semi-circular arc of radius R shown in Fig. 13-17.

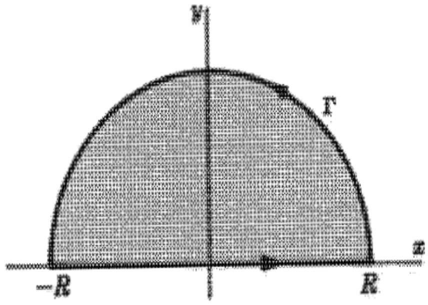

Fig. 13-17

By the result (4), page 287, we have

$$\left|\int_\Gamma f(z)\,dz\right| \leqq \int_\Gamma |f(z)|\,|dz| \leqq \frac{M}{R^k}\cdot \pi R = \frac{\pi M}{R^{k-1}}$$

since the length of arc $L = \pi R$. Then

$$\lim_{R\to\infty}\left|\int_\Gamma f(z)\,dz\right| = 0 \qquad \text{and so} \qquad \lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

13.29. Evaluate $\int_0^\infty \frac{dx}{x^4+1}$.

Consider $\oint_C \frac{dz}{z^4+1}$, where C is the closed contour of Problem 13.27 consisting of the line from $-R$ to R and the semi-circle Γ, traversed in the positive (counterclockwise) sense.

Since $z^4+1=0$ when $z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$, these are simple poles of $1/(z^4+1)$. Only the poles $e^{\pi i/4}$ and $e^{3\pi i/4}$ lie within C. Then using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/4} = \lim_{z \to e^{\pi i/4}} \left\{ (z - e^{\pi i/4}) \frac{1}{z^4+1} \right\}$$

$$= \lim_{z \to e^{\pi i/4}} \frac{1}{4z^3} = \frac{1}{4} e^{-3\pi i/4}$$

$$\text{Residue at } e^{3\pi i/4} = \lim_{z \to e^{3\pi i/4}} \left\{ (z - e^{3\pi i/4}) \frac{1}{z^4+1} \right\}$$

$$= \lim_{z \to e^{3\pi i/4}} \frac{1}{4z^3} = \frac{1}{4} e^{-9\pi i/4}$$

Thus

$$\oint_C \frac{dz}{z^4+1} = 2\pi i \{ \tfrac{1}{4} e^{-3\pi i/4} + \tfrac{1}{4} e^{-9\pi i/4} \} = \frac{\pi\sqrt{2}}{2} \tag{1}$$

i.e.

$$\int_{-R}^{R} \frac{dx}{x^4+1} + \int_\Gamma \frac{dz}{z^4+1} = \frac{\pi\sqrt{2}}{2} \tag{2}$$

Taking the limit of both sides of (2) as $R \to \infty$ and using the results of Problem 13.28, we have

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{2}$$

Since $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\int_0^{\infty} \frac{dx}{x^4+1}$, the required integral has the value $\dfrac{\pi\sqrt{2}}{4}$.

13.30. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

The poles of $\dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$ enclosed by the contour C of Problem 13.27 are $z = i$ of order 2 and $z = -1+i$ of order 1.

Residue at $z = i$ is $\displaystyle\lim_{z \to i} \frac{d}{dz} \left\{ (z-i)^2 \frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)} \right\} = \frac{9i-12}{100}$.

Residue at $z = -1+i$ is $\displaystyle\lim_{z \to -1+i} (z+1-i) \frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$.

Then

$$\oint_C \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i \left\{ \frac{9i-12}{100} + \frac{3-4i}{25} \right\} = \frac{7\pi}{50}$$

or

$$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero by Problem 13.27, we obtain the required result.

13.31. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta}$.

Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i}$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta} = \oint_C \frac{dz/iz}{5 + 3\left(\dfrac{z - z^{-1}}{2i}\right)} = \oint_C \frac{2\,dz}{3z^2 + 10iz - 3}$$

where C is the circle of unit radius with center at the origin, as shown in Fig. 13-18 below.

The poles of $\dfrac{2}{3z^2 + 10iz - 3}$ are the simple poles

$$\begin{aligned} z &= \frac{-10i \pm \sqrt{-100 + 36}}{6} \\ &= \frac{-10i \pm 8i}{6} \\ &= -3i,\ -i/3 \end{aligned}$$

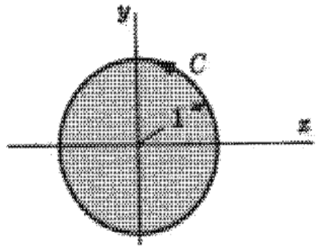

Fig. 13-18

Only $-i/3$ lies inside C.

Residue at $-i/3 = \displaystyle\lim_{z \to -i/3} \left(z + \frac{i}{3}\right)\left(\frac{2}{3z^2 + 10iz - 3}\right) = \lim_{z \to -i/3} \frac{2}{6z + 10i} = \frac{1}{4i}$ by L'Hospital's rule.

Then $\displaystyle\oint_C \frac{2\,dz}{3z^2 + 10iz - 3} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$, the required value.

13.32. Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta = \frac{\pi}{12}$.

If $z = e^{i\theta}$, $\cos\theta = \dfrac{z + z^{-1}}{2}$, $\cos 3\theta = \dfrac{e^{3i\theta} + e^{-3i\theta}}{2} = \dfrac{z^3 + z^{-3}}{2}$, $dz = iz\,d\theta$.

Then
$$\begin{aligned} \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta &= \oint_C \frac{(z^3 + z^{-3})/2}{5 - 4\left(\dfrac{z + z^{-1}}{2}\right)} \frac{dz}{iz} \\ &= -\frac{1}{2i}\oint_C \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\,dz \end{aligned}$$

where C is the contour of Problem 13.31.

The integrand has a pole of order 3 at $z = 0$ and a simple pole $z = \frac{1}{2}$ within C.

Residue at $z = 0$ is $\displaystyle\lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left\{z^3 \cdot \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\right\} = \frac{21}{8}$.

Residue at $z = \frac{1}{2}$ is $\displaystyle\lim_{z \to 1/2} \left\{(z - \tfrac{1}{2}) \cdot \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\right\} = -\frac{65}{24}$.

Then $\displaystyle -\frac{1}{2i}\oint_C \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\,dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8} - \frac{65}{24}\right\} = \frac{\pi}{12}$ as required.

13.34. Show that $\displaystyle\int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2}e^{-m},\ m > 0.$

Consider $\displaystyle\oint_C \frac{e^{imz}}{z^2+1}\,dz$ where C is the contour of Problem 13.27.

The integrand has simple poles at $z = \pm i$, but only $z = i$ lies within C.

Residue at $z = i$ is $\displaystyle\lim_{z\to i}\left\{(z-i)\frac{e^{imz}}{(z-i)(z+i)}\right\} = \frac{e^{-m}}{2i}.$

Then

$$\oint_C \frac{e^{imz}}{z^2+1}\,dz = 2\pi i\left(\frac{e^{-m}}{2i}\right) = \pi e^{-m}$$

or

$$\int_{-R}^{R} \frac{e^{imx}}{x^2+1}\,dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

i.e.

$$\int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}\,dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

and so

$$2\int_0^R \frac{\cos mx}{x^2+1}\,dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

Taking the limit as $R\to\infty$ and using Problem 13.33 to show that the integral around Γ approaches zero, we obtain the required result.

13.35. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

The method of Problem 13.34 leads us to consider the integral of e^{iz}/z around the contour of Problem 13.27. However, since $z = 0$ lies on this path of integration and since we cannot integrate through a singularity, we modify that contour by indenting the path at $z = 0$, as shown in Fig. 13-19, which we call contour C' or $ABDEFGHJA$.

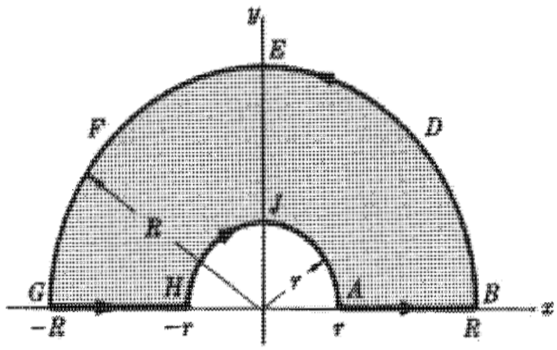

Fig. 13-19

Since $z = 0$ is outside C', we have

$$\int_{C'} \frac{e^{iz}}{z}\,dz = 0$$

or

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_r^R \frac{e^{ix}}{x}\,dx + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$$

Replacing x by $-x$ in the first integral and combining with the third integral, we find,

$$\int_r^R \frac{e^{ix}-e^{-ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$$

or

$$2i\int_r^R \frac{\sin x}{x}\,dx = -\int_{HJA} \frac{e^{iz}}{z}\,dz - \int_{BDEFG} \frac{e^{iz}}{z}\,dz$$

13.37. Show that $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1.$

Consider $\displaystyle\oint_C \frac{z^{p-1}}{1+z}\,dz$. Since $z = 0$ is a branch point, choose C as the contour of Fig. 13-22 where AB and GH are actually coincident with the x axis but are shown separated for visual purposes.

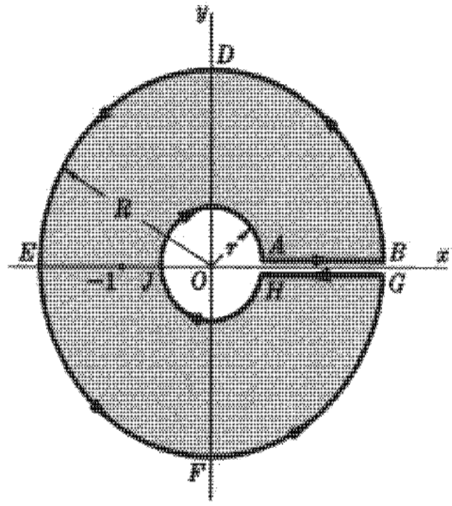

Fig. 13-22

The integrand has the pole $z = -1$ lying within C.

Residue at $z = -1 = e^{\pi i}$ is

$$\lim_{z\to -1}(z+1)\frac{z^{p-1}}{1+z} = (e^{\pi i})^{p-1} = e^{(p-1)\pi i}$$

Then $$\oint_C \frac{z^{p-1}}{1+z}\,dz = 2\pi i e^{(p-1)\pi i}$$

or, omitting the integrand,

$$\int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i e^{(p-1)\pi i}$$

We thus have

$$\int_r^R \frac{x^{p-1}}{1+x}\,dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1} iRe^{i\theta}\,d\theta}{1+Re^{i\theta}} + \int_R^r \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\,dx + \int_{2\pi}^0 \frac{(re^{i\theta})^{p-1} ire^{i\theta}\,d\theta}{1+re^{i\theta}} = 2\pi i e^{(p-1)\pi i}$$

where we have to use $z = xe^{2\pi i}$ for the integral along GH, since the argument of z is increased by 2π in going around the circle $BDEFG$.

Taking the limit as $r \to 0$ and $R \to \infty$ and noting that the second and fourth integrals approach zero, we find

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx + \int_\infty^0 \frac{e^{2\pi i(p-1)}x^{p-1}}{1+x}\,dx = 2\pi e^{(p-1)\pi i}$$

or $$(1 - e^{2\pi i(p-1)}) \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = 2\pi i e^{(p-1)\pi i}$$

so that $$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi i e^{(p-1)\pi i}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}} = \frac{\pi}{\sin p\pi}$$