Lecture # 9
Stirling Numbers

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1- Stirling Numbers of the First Kind

We explore a collection of combinatorial numbers which complement the Stirling numbers of the second kind, the Stirling numbers of the first kind, denoted $s(n,k)$. Stirling numbers of the first and second kind can be considered **opposite of each other**.

The easiest way to define $s(n,k)$ is by inverting the basis definition of for the Stirling numbers of the second kind (x^n in terms $\left\{\binom{x}{k}k!\right\}_{k=0}^{n}$):

$$x^n = \sum_{k=0}^{n}\binom{x}{k}k!S(n,k) \equiv \sum_{k=0}^{n}\binom{x}{k}k!\left\{ n \atop k \right\}.$$

We want to write $\binom{x}{n}n!$ in terms of the basis $\left\{x^k\right\}_{k=0}^{n}$. We do so by using $\left\{s(n,k)\right\}_{k=0}^{n}$ and say that.

$$n!\binom{x}{n} = \sum_{k=0}^{n} s(n,k)x^k . \qquad (1)$$

Formula (1) implies that $s(n,n)=1$, $s(n,k)=0$, for $k<0$ and $k \geq n$.

Table of values. Using the iteration method, we can build up the following table of values for the Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k)$. Such a table makes it easy to write a sum of factorial functions as an ordinary polynomial. The following is a table of values for the Stirling numbers of the first kind $\{s(n,k)\}_{k=0}^{n}$ where $0 \le k \le n$ and $0 \le n \ge 9$:

	$s(n,k)$									
n \ k	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	−1	1							
3	0	2	−3	1						
4	0	−6	11	−6	1					
5	0	24	−50	35	−10	1				
6	0	−120	274	−225	85	−15	1			
7	0	720	−1764	1624	−735	175	−21	1		
8	0	−5040	13068	−13132	6769	−1960	322	−28	1	
9	0	40320	−109584	118124	−67284	22449	−4536	546	−36	1

Table (1): Stirling numbers of the first kind

Recurrence relation. It is not hard to show that

$$\binom{x}{n+1}(n+1)! = x\binom{x}{n}n! - n\binom{x}{n}n!.$$

This identity when combined with formula (1) provides a two-term recurrence for $s(n+1,k)$. By definition (1) we have

$$(n+1)!\binom{x}{n+1} = \sum_{k=0}^{n+1} s(n+1,k)\,x^k .$$

On the other hand

$$(n+1)!\binom{x}{n+1} = x\binom{x}{n}n! - n\binom{x}{n}n! = x\sum_{k=0}^{n} s(n,k)x^k - n\sum_{k=0}^{n} s(n,k)x^k$$

$$= \sum_{k=0}^{n+1}\left[s(n,k-1) - ns(n,k)\right]x^k,$$

since $s(n,-1) = s(n,n+1) = 0$.

Comparing the coefficients of x^k gives us

$$s(n+1,k) = s(n,k-1) - ns(n,k). \qquad (2)$$

2- **<u>A combinatorial interpretation for</u>** $(-1)^{n-k} s(n,k)$ <u>in terms of</u> **<u>permutations</u>**.

The factor of $(-1)^{n-k}$ ensures that all the entries in table (1) are positive integers. Let n be a positive integer, and $[n] \equiv \{1,2,3,\ldots,n\}$.

A **permutation** of, n elements, $[n]$ is a map $\alpha : [n] \to [n]$ which is <u>one-to-one</u> and <u>onto</u>.

There are <u>three ways</u> to represent a permutation:

The **first** representation involves $2\times n$ array, where the first row is $1 \quad 2 \quad \ldots \quad n$ while the second row is the image $\alpha(1) \quad \alpha(2) \quad \ldots \quad \alpha(n)$.

The **second** representation for α is just the second row of the $2\times n$ array.

The **third** representation is the cycle notation of α.

A <u>cycle of a permutation</u> α is a nonempty ordered subset of $[n]$

given by $(a_1 \quad a_2 \quad \dots \quad a_j)$, where $a_{i+1} = \alpha(a_i)$ for $1 \le i \le j-1$ and, $\alpha(a_j) = a_1$. Because α is one-to-one and onto it can be decomposed into k disjoint cycles where $1 \le k \le n$.

The cycle representation is found by vertically tracing left to right through the $2 \times n$ array.

Example (1). We explain the three representations for the $3!$ permutations of $[3]$ in table (2):

Array representation	Row representation	Cycle representation
1 2 3 1 2 3	1 2 3	$(1)\ (2)\ (3)$
1 2 3 1 3 2	1 3 2	$(1)\ (23)$
1 2 3 2 1 3	2 1 3	$(12)\ (3)$
1 2 3 2 3 1	2 3 1	(123)
1 2 3 3 1 2	3 1 2	(132)
1 2 3 3 2 1	3 2 1	$(13)\ (2)$

Table (2) permutations of $[3]$

Unsigned Stirling numbers of the first kind

Let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the number of permutations of, n elements, $[n]$, with exactly k disjoint (or non-empty) cycles. This number is called a sign-less (or unsigned) Stirling numbers of the first kind. It count the number of permutations of, n elements, $[n]$, with exactly k disjoint cycles. The notation $s(n,k)$ is sometimes used for the unsigned Stirling numbers of the first kind. Furthermore define $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$. Clearly

$$\begin{bmatrix} n \\ k \end{bmatrix} = 0 \quad \text{if} \quad k > n \text{ or } k \le 0, \begin{bmatrix} n \\ k \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} n \\ n \end{bmatrix} = 1, \text{ and} \begin{bmatrix} n \\ 0 \end{bmatrix} = 0 \;\forall\, n.$$

Example (2). For the 4 element set $\{a,b,c,d\}$, there are $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11$ permutations containing exactly 2 non-empty cycles. They are

$$\begin{pmatrix} 1234 \\ 2314 \end{pmatrix} = (123)(4), \quad \begin{pmatrix} 1234 \\ 3124 \end{pmatrix} = (132)(4), \quad \begin{pmatrix} 1234 \\ 3241 \end{pmatrix} = (134)(2),$$

$$\begin{pmatrix} 1234 \\ 4213 \end{pmatrix} = (134)(2), \quad \begin{pmatrix} 1234 \\ 2431 \end{pmatrix} = (124)(3), \quad \begin{pmatrix} 1234 \\ 4132 \end{pmatrix} = (142)(3),$$

$$\begin{pmatrix} 1234 \\ 1342 \end{pmatrix} = (234)(1), \quad \begin{pmatrix} 1234 \\ 1423 \end{pmatrix} = (243)(1), \quad \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} = (12)(34),$$

$$\begin{pmatrix} 1234 \\ 3412 \end{pmatrix} = (13)(24), \quad \begin{pmatrix} 1234 \\ 4321 \end{pmatrix} = (14)(23).$$

Lemma (1). The Stirling numbers obey the recurrence relation:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} - n \begin{bmatrix} n \\ k \end{bmatrix}, \qquad (3)$$

The left side of equation (3) counts the permutations of $[n+1]$ with exactly k cycles. We require that the right side of equation (3) also counts this quantity.

Let α be a permutation of $[n+1]$ with exactly k cycles. Either α has a cycle of the form $(n+1)$ or $n+1$ is in a cycle of length greater than 1, i.e. $n+1$is part of a cycle which contains at least 2 elements.

If $(n+1)$ is an independent cycle the rest of α is a permutation of $[n]$ with $k-1$ cycles. Such permutations are counted by $\begin{bmatrix} n \\ k-1 \end{bmatrix}$.

Suppose that $n+1$ is not isolated. Then it must belong to one of the cycles of α_2, where α_2 is a permutation of $[n]$ with k cycles. Represent α_2 as

$$(a_{11} \quad a_{12} \quad \ldots \quad a_{1j_1}) \quad (a_{21} \quad a_{22} \quad \ldots \quad a_{2j_2}) \quad \ldots \quad (a_{k1} \quad a_{k2} \quad \ldots \quad a_{kj_k})$$

where $\sum_{i=1}^{k} j_i = n$ working from left to right there are n ways to insert $n+1$ into these k cycles, namely by placing it immediately to the right one of the a_{ij_i} digits in the cycle structure.

The rule of products implies there are $n\begin{bmatrix} n \\ k \end{bmatrix}$such permutations of $[n+1]$ which have k cycles, none of which have the form $(n+1)$.

Applying the rule of sums, we obtain the right side of formula (3).

Assume that the sign-less Stirling numbers of the first kind “or Stirling cycle number” is $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k)$.

Multiply the two sides of equation (2) by $(-1)^{n+1-k}$

$$(-1)^{n+1-k} s(n+1,k) = (-1)^{n+1-k} s(n,k-1) + (-1)^{n-k} ns(n,k). \quad (4)$$

Equation (4) is exactly equation (3) with the substitution

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k)\cdot$$

In other words, $\left\{\begin{bmatrix} n \\ k \end{bmatrix}\right\}_{n=0}^{\infty}$ and $\left\{(-1)^{n-k} s(n,k)\right\}_{n=0}^{\infty}$ obey the same recurrence relation and share the same initial conditions.

We use the equivalence of the recurrence relations to conclude that

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k), \ \forall \text{ nonnegative integers } n \text{ and } k.$$

Def (1). Stirling numbers of the first kind)

Stirling numbers of the first kind, $s(n,k)$, are the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \text{, for } n = 1,2,... \quad (5)$$

where $(x)_n$ is the falling factorial: $(x)_0 = 1$ and

$$(x)_n := x(x-1)(x-2)...(x-n+1) = n!\binom{x}{n}.$$

For example:

$$(x)_3 = x(x-1)(x-2) = 2x - 3x^2 + x^3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} x - \begin{bmatrix} 3 \\ 2 \end{bmatrix} x^2 + \begin{bmatrix} 3 \\ 3 \end{bmatrix} x^3.$$

From formula (5), we have

$$(x)_n = x^n + \begin{bmatrix} n \\ n-1 \end{bmatrix} x^{n-1} + \ldots + \begin{bmatrix} n \\ 2 \end{bmatrix} x^2 + \begin{bmatrix} n \\ 1 \end{bmatrix} x,$$

$$x(x)_n = x^{n+1} + \begin{bmatrix} n \\ n-1 \end{bmatrix} x^n + \begin{bmatrix} n \\ n-2 \end{bmatrix} x^{n-1} + \ldots + \begin{bmatrix} n \\ 2 \end{bmatrix} x^2 + \begin{bmatrix} n \\ 1 \end{bmatrix} x^2,$$

$$n(x)_n = nx^n + n\begin{bmatrix} n \\ n-1 \end{bmatrix} x^{n-1} + \ldots + n\begin{bmatrix} n \\ 2 \end{bmatrix} x^2 + n\begin{bmatrix} n \\ 1 \end{bmatrix} x.$$

Thus, subtracting corresponding members of the last two equations

$$(x)_n (x-n) = x^{n+1} + \sum_{k=0}^{n} \left(\begin{bmatrix} n \\ k-1 \end{bmatrix} - n \begin{bmatrix} n \\ k \end{bmatrix} \right) x^k \text{, taking } \begin{bmatrix} n \\ -1 \end{bmatrix} = 0$$

$$= \sum_{k=0}^{n+1} \left(\begin{bmatrix} n \\ k-1 \end{bmatrix} - n \begin{bmatrix} n \\ k \end{bmatrix} \right) x^k, \qquad \text{taking } \begin{bmatrix} n \\ n+1 \end{bmatrix} = 0.$$

But $(x)_{n+1} = x(x-1)(x-2)\ldots(x-n+1)(x-n) = (x)_n (x-n)$, and from (5)

$$(x)_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} s(n+1,k) x^k = \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix} x^k.$$

Consequently, the Stirling numbers obey the recurrence relation (3) with the initial conditions $\begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{n0}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$ where δ_{n0} is the Kronecker delta.

<u>Example.</u> $(x)_5 + 4(x)_4 + 3(x)_3$

$$= x^5 + [-10 + 4(1)] x^4 + (35 + 4(-6) + 3(1)) x^3$$

$$+\left[-50+4(11)+3(-3)\right]x^2+\left[24+4(-6)+3(2)\right]x$$

$$=x^5-6x^4+14x^3-15x^2+6x.$$

Stirling & Pascal Matrices

Def (2). The **Stirling matrix** of the second kind $\mathbf{S}_n=\left(S_n(i,j)\right)_{1\le i,j\le n}$ and the **Stirling matrix** of the first kind $\mathbf{s}_n=\left(s_n(i,j)\right)_{1\le i,j\le n}$, are defined, respectively, by

$$S_n(i,j)=\begin{cases}S(i,j), \text{ if } 1\le j\le i\le n\\ 0, \qquad \text{otherwise}\end{cases} \text{ and } s_n(i,j)=\begin{cases}s(i,j), \text{ if } 1\le j\le i\le n\\ 0, \qquad \text{otherwise}\end{cases},$$

where $s(i,j)$ is the element in the i^{th} row and j^{th} column:

$$s(i,j)=s(i-1,j-1)-(i-1)s(i-1,j).$$

For a non-negative integer n, the Stirling matrix of the **first** kind $\mathbf{S}_n=\left(S(i,j)\right)$ for $i=0,1,2,\ldots,n$ and $j=0,1,2,\ldots,i$ satisfies

$$\begin{pmatrix} s(0,0) & 0 & 0 & \cdots & 0\\ s(1,0) & s(1,1) & 0 & \cdots & 0\\ s(2,0) & s(2,1) & s(2,2) & & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ s(n,0) & s(n,1) & \cdots & s(n,n-1) & s(n,n)\end{pmatrix}\begin{pmatrix}x^0\\ x^1\\ x^2\\ \vdots\\ x^n\end{pmatrix}=\begin{pmatrix}(x)_0\\ (x)_1\\ \\ \vdots\\ (x)_n\end{pmatrix}, \quad (1)$$

and the Stirling matrix of the **second** kind $\mathbf{S}_n=\left(S(i,j)\right)$ for $i=0,1,2,\ldots,n$ and $j=0,1,2,\ldots,i$ satisfies

$$\begin{pmatrix} S(0,0) & 0 & 0 & \cdots & 0 \\ S(1,0) & S(1,1) & 0 & \cdots & 0 \\ S(2,0) & S(2,1) & S(2,2) & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S(n,0) & S(n,1) & \cdots & S(n,n-1) & S(n,n) \end{pmatrix} \begin{pmatrix} (x)_0 \\ (x)_1 \\ \\ \vdots \\ (x)_n \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}. \quad (2)$$

For example, for $n=5$

$$\mathbf{S}_5 = \left(S_n(i,j)\right)_{1\le i,j\le 5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{pmatrix}, \text{ and}$$

$$\mathbf{s}_5 = \left(s_n(i,j)\right)_{1\le i,j\le 5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{pmatrix}.$$

In other words, the Stirling matrices of the **first** and second kinds, $\mathbf{s}_n$ and $\mathbf{S}_n$ are **inverse** to each other, i.e.,

$$\mathbf{S}_n \mathbf{s}_n = \mathbf{I}_n \Rightarrow \mathbf{S}_n^{-1} = \mathbf{s}_n,$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

Def (3) (Pascal type matrices)

The Pascal type matrices $\mathbf{P} = \left(p(i,j)\right)_{1\le i,j\le n}$, $\overline{\mathbf{P}} = \left(\overline{p}(i,j)\right)_{0\le i,j\le n}$, and

$\widehat{\mathbf{P}} = \left(\widehat{p}(i,j)\right)_{0\le i,j\le n}$ are $(n+1)\times(n+1)$ matrices which are defined with the binomial coefficients by (the lower triangular array):

$$p(i,j)=\begin{cases}\binom{i-1}{j-1}, & \text{if } 1\le j\le i\\ 0, & \text{otherwise}\end{cases},\quad \overline{p}(i,j)=\begin{cases}\binom{i}{j}, & \text{if } 1\le j\le i\\ 0, & \text{otherwise}\end{cases}, \text{ and}$$

$$\widehat{p}(i,j)=\begin{cases}\binom{i}{j-1}, & \text{if } 1\le j\le i\\ 0, & \text{otherwise}\end{cases}.$$

The matrix $\mathbf{P}$ is the Pascal matrix, and $\overline{\mathbf{P}}$ is the Pascal 1- eliminated matrix which is obtained from the Pascal matrix by deleting its first row and column, while and $\widehat{\mathbf{P}}$ is the "reverse" of $\overline{\mathbf{P}}$.

It is easy to check that, $\mathbf{P}^{-1}=\mathbf{JPJ}$, and $\overline{\mathbf{P}}^{-1}=\mathbf{J}\overline{\mathbf{P}}\mathbf{J}$, where $\mathbf{J}=\operatorname{diag}\left(1,-1,\ldots,(-1)^{n+1}\right)$.

Let $\tilde{\mathbf{P}}=\mathbf{J}\widehat{\mathbf{P}}\mathbf{J}$ we have $\tilde{\mathbf{P}}=\overline{\mathbf{P}}^{-1}\widehat{\mathbf{P}}=\widehat{\mathbf{P}}\mathbf{P}^{-1}$ and $\tilde{\mathbf{P}}^{-1}=\mathbf{J}\widehat{\mathbf{P}}^{-1}\mathbf{J}=\widehat{\mathbf{P}}^{-1}\overline{\mathbf{P}}=\mathbf{P}\widehat{\mathbf{P}}^{-1}$.

Lemma (2). Let $\mathbf{\Lambda}=\operatorname{diag}(1,2,\ldots,n)$ be a diagonal matrix, then the Pascal type matrix $\tilde{\mathbf{P}}$ can be factorized into the products of the Stirling matrices $\tilde{\mathbf{P}}=\mathbf{S}_n\mathbf{\Lambda}\mathbf{s}_n$.

Using the following Pascal type matrix

$$\left(\tilde{\mathbf{P}}\right)_{i,j}=\begin{cases}(-1)^{i-j}\binom{i}{i-j+1}, & \text{if } i\ge j,\\ 0, & \text{otherwise}\end{cases}$$

Define the matrix $\mathbf{\Lambda}$ of the eigenvalues of the matrix $\tilde{\mathbf{P}}$, in the case

$1 \le j \le i \le 5$, then

$$\tilde{\mathbf{P}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 \\ 1 & -5 & 10 & -10 & 5 \end{pmatrix}, \text{ and } \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

Thus, $\tilde{\mathbf{P}}\mathbf{S}_n = \mathbf{S}_n\mathbf{\Lambda} \rightarrow \tilde{\mathbf{P}} = \mathbf{S}_n\mathbf{\Lambda}\mathbf{s}_n$, or $\tilde{\mathbf{P}}^{-1} = \mathbf{S}_n\mathbf{\Lambda}^{-1}\mathbf{s}_n$, with

$$\mathbf{S}_n = \tilde{\mathbf{P}}^{-1}\mathbf{s}_n^{-1}\mathbf{\Lambda} \quad \text{and} \quad \mathbf{s}_n = \mathbf{\Lambda}\mathbf{S}_n^{-1}\tilde{\mathbf{P}}^{-1}$$

Therefore,

$$\tilde{\mathbf{P}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}.$$