Example (1). Using the first step-decomposition theorem (1) given in the previous Lecture # 7. Let $\{X_n, n=0,1,...\}$ be a MC with TPM $\mathbf{M}=\left(p_{ij}\right)$, on the state space $SS=\{0,1\}$. we can easily establish the following:

$$f_{00}^{(1)}=\Pr\left(T_{00}=1\right)=\Pr\left(X_1=0 \middle| X_0=0\right)=p_{00},$$

$$f_{00}^{(n)}=\Pr\left(T_{00}=n\right)=\sum_{k\in SS-\{0\}} p_{0k}\, f_{k0}^{(n-1)}=p_{01}\, f_{10}^{(n-1)}=p_{01}p_{11}\, f_{10}^{(n-2)}$$

$$=\ldots=p_{01}\left(p_{11}\right)^{n-2} p_{10},\; n\geq 2,$$

Similarity, $f_{11}^{(n)}=\begin{cases} p_{11}, & n=1 \\ p_{10}\left(p_{00}\right)^{n-2} p_{01}, & n\geq 2 \end{cases},$

$$f_{01}^{(n)}=\left(p_{00}\right)^{n-1} p_{01}, \text{ and } f_{10}^{(n)}=\left(p_{11}\right)^{n-1} p_{10}, \text{ for } n\geq 1.$$

Bernoulli Trials:

Consider the tossing of an unfair coin with success given by $\{\text{head}=s\}$ with probability p and failure given by $\{\text{tail}=f\}$ with probability q, and $p+q=1$. The sample space is $S_1=\{s,f\}$. If the coin is tossed twice, then the sample space $S_2=\{ss, sf, fs, ff\}$ is an ordered set from the Cartesian product $S_1\times S_1$. The cardinality of S_2 is $2^2=4$. The probability of two successes is p^2 and two failures is q^2. If we toss the coin n times, then the resulting sample space is $S_n=S_1\times S_1\times\ldots\times S_1$ and the cardinality of S_n is 2^n. The probability of n successes is p^n.

Def. (Bernoulli trials)

Bernoulli trials are repeated functionally independent trials with only two events s and f for each trial. The trials are also statistically independent with the two events s and f in each trial having probabilities p and $q=1-p$, respectively. These are called independent identically distributed (iid) trials.

Def. (Bernoulli process)

If X_n is a random variable denotes the number of successes in the trial n, The stochastic process $\{X_n, n=1,2,...\}$ is called Bernoulli process with probability of success $p\,(0\le p\le 1)$, if it satisfies:

1- The random variables $X_1, X_2, \cdots$ are independent

2- The event $\{X_n=1\}$ denotes to the fail in the trial number n, while the event $\{X_n=0\}$ denotes to the success in the trial n, i.e.,

$$X_n=\begin{cases}1 & \text{for success } s \text{ in the } n^{th} \text{trial}\\ 0 & \text{for failure } f \text{ in the } n^{th} \text{trial}\end{cases},$$

with probabilities $\Pr(X_n=1)=p$, and $\Pr(X_n=0)=q=1-p\ \forall\ n=1,2,\ldots$

In the Bernoulli process the parameter set is $T=\{1,2,...\}$ and the state space is $SS=\{0,1\}$ the two are discrete.

The statistics of Bernoulli process

Mean: $E[X_n]=\mu_{X_n}=1.\Pr(X_n=1)+0.\Pr(X_n=0)=p$

Second moment: $E\left[X_n^2\right]=\mu_{X_n^2}=1^2.\Pr(X_n=1)+0^2.\Pr(X_n=0)=p$

Variance: $Var(X_n)=E\left[X_n^2\right]-\left(E[X_n]\right)^2=p-p^2=p(1-p)$

Probability generating function

$$G_X(z)=E\left[z^{X_n}\right]=z^0\times\Pr(X_n=0)+z^1\times\Pr(X_n=1)=q+zp$$

Autocorrelation function:

$$R_X(m,n) = E[X_m X_n] = \begin{cases} 1^2.p + 0^2.(1-p) = p, & m = n \\ E[X_m]E[X_n] = p^2, & m \neq n \end{cases}$$

Auto-covariance:

$$Cov(X_m, X_n) = R_X(m,n) - E[X_m]E[X_n] = \begin{cases} p - p^2 = p(1-p), & m = n \\ p^2 - p^2 = 0, & m \neq n \end{cases}$$

Normalized auto-covariance: $\rho_X(m,n) = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$

Since the mean is independent of time n and the autocorrelation depends only on the time difference $m-n$, the process X_n is stationary.

Def. (Binomial process)

If $\{X_n,\ n = 1,2,...\}$ represents Bernoulli process with probability of success, $p\,(0 \le p \le 1)$, and N_n represents the number of successes during the first trials until the completion of the trial n:

$$N_n = \begin{cases} X_1 + X_2 + \cdots + X_n, & n \ge 1 \\ 0, & n = 0 \end{cases},$$

where the increments $\{X_i\}$ form a family of independent $\{0,1\}$-valued random variables.

The process $\{N_n,\ n = 1,2,...\}$ is called a binomial process.

The statistics of the binomial process

Mean: Since the probability of each success is p, the mean value of N_n can be obtained using the linearity of the expectation operator:

$$\mu_{N_n}=E[N_n]=E\left[\sum_{i=1}^{n}X_i\right]=\sum_{i=1}^{n}E[X_i]=nE[X_i]=np$$

Second moment: The second moment of N_n can be given by

$$E[N_n^2]=E\left[\left(\sum_{i=1}^{n}X_i\right)^2\right]=\sum_{i=1}^{n}E[X_i^2]+\sum_{i\neq j}\sum_{j}E[X_i]E[X_j]=np+n(n-1)p^2$$

Variance:

$$Var(N_n)=E[N_n^2]-(E[N_n])^2=np+n(n-1)p^2-n^2p^2=np(1-p).$$

Probability generating function

$$G_{N_n}(z)=E[z^{N_n}]=E[z^{X_1+X_2+...X_n}]=E[z^{X_1}]E[z^{X_2}]...E[z^{X_n}]=(q+zp)^n$$

Autocorrelation:

$$R_N(m,n)=E[N_mN_n]=E\left[\sum_{j=1}^{n}\sum_{i=1}^{m}X_iX_j\right]$$

$$=E\left[\sum_{i=1}^{m}X_i^2\right]+\sum_{j=1}^{m}\sum_{\substack{i=1\\ i\neq j}}^{m}E[X_iX_j]+\sum_{j=1}^{n-m}\sum_{\substack{i=1\\ i\neq j}}^{m}E[X_iX_j]$$

$$=mp+m(m-1)p^2+m(n-m)p^2=\begin{cases}mp(1-p)+mnp^2 & \text{for } m\le n\\ np(1-p)+mnp^2 & \text{for } n\le m\end{cases}$$

$$=p(1-p)\min(m,n)+mnp^2,\quad m,n>0.$$

Auto-covariance:

$$Cov(N_m,N_n)=R_N(m,n)-E[N_m]E[N_n]=p(1-p)\min(m,n),\quad m,n>0.$$

Binomial process is a Markov chain

If the random variable N_n denotes the number of successes during the number n of Bernoulli's trials, where the probability of success in any one trial is p, the sequence of $\{N_n,\ n=1,2,...\}$ is a MC, the probability of transition in one step is

$$p_{ij} = \Pr\left(N_{n+1} = j \mid N_n = i\right) = \Pr\left(X_{n+1} = j - i\right) = \begin{cases} p, & j = i+1, \\ 1-p, & j = i, \\ 0, & \text{otherwise} \end{cases},$$

and the transition probabilities after n-step is given by

$$p_{ij}^{(n)} = \binom{n}{j-i} p^{j-i} q^{n-j+1}, \; j = i, \ldots, n+i \cdot$$

If N_n is the number of successes in the first n Bernoulli trials, with probability of a success in any one trial is p, then the sequence of random variables $\{N_n, n = 1, 2, ...\}$ is a MC, with $N_0 = 0$, since the process $\{N_n\}$ has the discrete parameter set $T = \{1, 2, ...\}$ and the discrete state space $SS = \{0, 1, 2, ...\}$, and satisfies the Markov property: for all $i, k, i_1, ..., i_{n-1} \in SS$ we have

$$\Pr\left(N_{n+1} = k \middle| N_n = i, N_{n-1} = i_{n-1}, \ldots, N_1 = i_1\right)$$

$$= \frac{\Pr\left(N_{n+1} = k, N_n = i, N_{n-1} = i_{n-1}, \ldots, N_1 = i_1\right)}{\Pr\left(N_n = i, N_{n-1} = i_{n-1}, \ldots, N_1 = i_1\right)}$$

$$= \frac{\Pr\left(N_{n+1} - N_n = k - i, N_n - N_{n-1} = i - i_{n-1}, ..., N_2 - N_1 = i_2 - i_1, N_1 = i_1\right)}{\Pr\left(N_n - N_{n-1} = i - i_{n-1}, ..., N_2 - N_1 = i_2 - i_1, N_1 = i_1\right)}$$

$$= \frac{\Pr\left(X_{n+1} = k - i, X_n = i - i_{n-1}, ..., X_1 = i_2 - i_1, N_1 = i_1\right)}{\Pr\left(X_n = i - i_{n-1}, X_{n-1} = i_{n-1} - i_{n-2}, ..., X_1 = i_2 - i_1, N_1 = i_1\right)}$$

$$= \Pr\left(X_{n+1} = k - i \middle| X_n = i - i_{n-1}, ..., X_1 = i_2 - i_1, N_1 = i_1\right)$$

$$= \Pr\left(X_{n+1} = k - i \middle| N_n = i, N_{n-1} = i_{n-1}, \ldots, N_1 = i_1\right)$$

$$= \Pr\left(X_{n+1} = k - i \middle| \sum_{i=1}^{n} X_i = i, \sum_{i=1}^{n-1} X_i = i_{n-1}, \ldots, N_1 = i_1\right)$$

$$= \Pr\left(X_{n+1} = k - i\right) = \Pr\left(X_{n+1} = k - i\right) \Pr\left(\sum_{i=1}^{n} X_i = i\right) / \Pr\left(\sum_{i=1}^{n} X_i = i\right)$$

$$= \Pr\left(X_{n+1} = k - i, \sum_{i=1}^{n} X_i = i\right) / \Pr\left(\sum_{i=1}^{n} X_i = i\right) = \Pr\left(X_{n+1} = k - i \middle| \sum_{i=1}^{n} X_i = i\right)$$

$$= \Pr\left(X_{n+1} = k - i \middle| N_n = i\right) = \Pr\left(N_{n+1} = k \middle| N_n = i\right),$$

where the third equality is due to the independence of X_{n+1} and the other n random variables. Thus, the future of a process depends only on the most recent past outcome.

The probability $\Pr\left(N_{n+1} = k \middle| N_n = i, N_{n-1} = i_{n-1}, \ldots, N_1 = i_1\right)$ depends on the value of N_n and is independent of the values of $N_1, N_2, ..., N_{n-1}$, since $N_{n+1} = N_n + X_{n+1}$, i.e.,

$$\Pr\left(N_{n+1} = j \middle| N_n = i\right) = p, \qquad \Pr\left(N_{n+1} = i \middle| N_n = i\right) = 1 - p.$$

So, N_{n+1} depends only on N_n, and both the state space and the parameter set are discrete, then the process $\{N_n, n = 1, 2, ...\}$ is an example of MC, on the state space $SS = \{0, 1, 2, ...\}$, with parameter set $T = \{1, 2, ...\}$ and one-step transition probability:

$$p_{ij} = \Pr\left(N_{n+1} = j \mid N_n = i\right) = \Pr\left(X_{n+1} = j - i\right) = \begin{cases} p, & j = i + 1, \\ 1 - p, & j = i, \\ 0, & \text{otherwise} \end{cases}$$

The TPM is $$\mathbf{M} = \left(p_{ij}\right)_{i, j \in SS} = \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ \\ \end{matrix} \begin{pmatrix} 1-p & p & 0 & & \cdots \\ 0 & 1-p & p & 0 & \cdots \\ 0 & 0 & 1-p & p & \\ \vdots & & & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & \end{pmatrix}.$$

In addition, the MC $\{N_n\}_{n\in\mathbb{N}}$ is <u>time</u> <u>homogeneous</u> if the random sequence $\{X_n\}_{n\geq 1}$ is identically distributed.

Lemma (1) (Probability distribution)

The number N_n of successes in the first n Bernoulli trials, with probability of a success in any one trial is p, is a binomial random variable:

$$\Pr\left(k \text{ successes in any sequence in } n \text{ trials}\right)$$

$$= \Pr\left(N_n = k\right) = \binom{n}{k} p^k q^{n-k}, \text{ for } p+q=1,\ k=0,1,...,n. \quad \text{(i)}$$

Proof. The state probability at time $n+1$ can be determined from the relation: $p_k^{(n+1)} = \sum_i p_i^{(n)} p_{ik}$, i.e.,

$$p_k^{(n+1)} = \Pr\left(N_{n+1} = k\right) = \sum_i \Pr\left(N_n = i\right) \Pr\left(N_{n+1} = k \middle| N_n = i\right),$$

since $\Pr\left(N_{n+1} = k \middle| N_n = i\right) = \Pr\left(X_{n+1} = N_{n+1} - N_n = k - i\right) = \begin{cases} q, & i = k \\ p, & i = k-1 \\ 0, & \text{otherwise} \end{cases}$,

the following recurrence relation follows

$$p_k^{(n+1)} = \Pr\left(N_{n+1} = k\right) = p\Pr\left(N_n = k-1\right) + q\Pr\left(N_n = k\right). \quad \text{(ii)}$$

Using the mathematical induction with the recurrence relation (ii), we will prove formula (i):

For $n=0$, formula (i) is true, since $N_0 = 0$.

Assume that it is also true for $n=m$ and all k. That is

$$\Pr(N_m = k) = \binom{m}{k} p^k q^{m-k}, \text{ for } k = 0,1,\ldots,m. \qquad \text{(iii)}$$

We will prove it is true for $n=m+1$, i.e.,

$$\Pr(N_{m+1} = k) = \binom{m+1}{k} p^k q^{m+1-k}. \qquad \text{(iv)}$$

Putting formula (iii) in the recurrence relation (ii), we get

L.H.S. $\Pr(N_{m+1} = k) = p\Pr(N_m = k-1) + q\Pr(N_m = k)$

$$= p\binom{m}{k-1} p^{k-1} q^{m-k+1} + q\binom{m}{k} p^k q^{m-k}$$

$$= \binom{m}{k-1} p^k q^{m-k+1} + \binom{m}{k} p^k q^{m-k+1} = \left[\binom{m}{k-1} + \binom{m}{k}\right] p^k q^{m-k+1}$$

$$= \binom{m+1}{k} p^k q^{m-k+1} = \text{R.H.S., for } 0 < k \le m+1$$

when $k=0$, we get $\Pr(N_{m+1} = 0) = p \times 0 + q\Pr(N_m = 0) = q \times q^m = q^{m+1}$

Therefore, the probability of k successes in n trials is given by a binomial distribution: $\Pr(N_n = k) = \binom{n}{k} p^k q^{n-k}$, for $k = 0,1,\ldots,n$.

The n^{th}-step transition probabilities

$$p_{ij}^{(n)} = \Pr(N_{m+n} = j \mid N_m = i) = \Pr\left(\sum_{k=1}^{m+n} X_k = j \mid \sum_{k=1}^{m} X_k = i\right)$$

$$= \Pr\left(\sum_{k=m+1}^{m+n} X_k = j - i\right) = \Pr(N_n = j - i)$$

$$=\begin{cases}\binom{n}{j-i} p^{j-i}\, q^{n-(j-i)}, & j=i,i+1\ldots,i+n; \quad p+q=1\\ 0, & j<i\end{cases}$$

<u>The n^{th}-step TPM</u>

$$\mathbf{M}^{(n)}=\left(p_{ij}^{(n)}\right)_{i,j\in SS}=\begin{pmatrix} q^n & npq^{n-1} & \binom{n}{2}p^2q^{n-2} & \binom{n}{3}p^3q^{n-3} & \cdots & \binom{n}{j}p^jq^{n-j} & \cdots \\ 0 & q^n & npq^{n-1} & \binom{n}{2}p^2q^{n-2} & \ddots & \binom{n}{j-1}p^{j-1}q^{n-(j-1)} & \cdots \\ 0 & 0 & q^n & npq^{n-1} & \ddots & & \\ 0 & & 0 & q^n & \ddots & & \vdots \\ 0 & & & 0 & \ddots & & \\ & & & & 0 & q^n & \\ \vdots & 0 & \cdots & & \cdots & 0 & \ddots \end{pmatrix}.$$

Unlike the Bernoulli process, the mean of the binomial process is dependent on time n and the autocorrelation function is dependent on both m, n and hence is a **non-stationary** process.

<u>Note that</u> the increment $N_{n+m}-N_m$ represents the number of successes through the trials $m+1, m+2,\ \ldots, m+n$:

$$N_{m+n}-N_m=\sum_{k=1}^{m+n}X_k-\sum_{k=1}^{m}X_k=\sum_{k=m+1}^{m+n}X_k=X_{m+1}+X_{m+2}+\cdots+X_{m+n}.$$

It is also a sum of n independent random variables that have the same distribution of the Bernoulli distribution, from which we conclude that

$$\Pr\left(N_{m+n}-N_m=j\right)=\Pr\left(N_n=j\right)=\binom{n}{j}p^j(1-p)^{n-j},\quad j=0,1,\ldots,n.$$

It is the element number j in the binomial expansion $(p+q)^n$ and it does not depend on m where $q=1-p$, for example

$$\Pr(N_6 = 3) = \binom{6}{3} p^3 (1-p)^{6-3} = 20\, p^3 (1-p)^3$$

$$\Pr(N_{15} - N_{10} = 4) = \Pr(N_5 = 4) = \binom{5}{4} p^4 (1-p)^{5-4} = 5\, p^4 (1-p).$$

<u>Lemma</u> <u>(2)</u>: The conditional probability of the number of successes, $N_{n+m} - N_m$ during number of trials $m+1, m+2, \ldots, m+n$, is <u>independent</u> of the number of previous successes until trial number m "which is $N_1, N_2, \ldots, N_m$", for $k = 0,1,\ldots,n$

$$\Pr\left(N_{m+n} - N_m = k \mid N_m = k_m, N_{m-1} = k_{m-1}, .., N_1 = k_1\right)$$

$$= \Pr(N_{m+n} - N_m = k) = \Pr(N_n = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Proof. Using the definition of a random variable N_n, we find that the random variables $N_1, N_2, \ldots, N_m$ are exact determined in terms of $X_1, X_2, \ldots, X_m$ and vice versa, i.e., $\{N_1, N_2, \ldots, N_m\} \Leftrightarrow \{X_1, X_2, \ldots, X_m\}$:

$$X_1 = N_1, X_2 = N_2 - N_1, \ldots, X_m = N_m - N_{m-1}.$$

Thus, $\Pr\left(N_{m+n} - N_m = k \mid N_m = k_m, N_{m-1} = k_{m-1}, .., N_1 = k_1\right)$

$$= \Pr\left(N_{m+n} - N_m = k \mid X_m = x_m, X_{m-1} = x_{m-1}, .., X_1 = x_1\right),$$

(with $x_i = k_i - k_{i-1}$, for $i = 1,2,\ldots$)

$$= \Pr\left(X_{m+1} + X_{m+2} + \ldots + X_{m+n} = k \mid X_m = x_m, X_{m-1} = x_{m-1}, .., X_1 = x_1\right)$$

In other meaning $N_{m+n} - N_m = X_{m+1} + X_{m+2} + \ldots + X_{m+n}$ and $\{X_{m+1}, X_{m+2}, \ldots, X_{m+n}\}$ are independent of $\{X_m, X_{m-1}, .., X_1\}$. Thus

$$\Pr\left(N_{m+n} - N_m = k \mid X_m = x_m, X_{m-1} = x_{m-1}, .., X_1 = x_1\right)$$

$$= \Pr(N_{m+n} - N_m = k) = \Pr(N_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0,1,\ldots,n$$

Corollary (1). If $n_j > \cdots > n_2 > n_1 > n_0 = 0$ are positive integers, then the random variables (increments) $N_{n_j} - N_{n_{j-1}}, \ldots, N_{n_2} - N_{n_1}, N_{n_1} - N_{n_0}$ are independent.

Example (1). Find the following

- the JPMF $\Pr\left(N_{13} = 8, N_7 = 5, N_5 = 4\right)$
- the expected value $E\left[N_5 N_8\right]$.

Solution. Since the following two events are equivalent

$$\left\{N_{13} = 8, N_7 = 5, N_5 = 4\right\} \text{ and } \left\{N_5 = 4, N_7 - N_5 = 1, N_{13} - N_7 = 3\right\},$$

then $\Pr\left(N_{13} = 8, N_7 = 5, N_5 = 4\right) = \Pr\left(N_5 = 4, N_7 - N_5 = 1, N_{13} - N_7 = 3\right)$

$$= \Pr\left(N_{13} - N_7 = 3 \middle| N_5 = 4, N_7 - N_5 = 1\right) \Pr\left(N_5 = 4, N_7 - N_5 = 1\right)$$

$$= \Pr\left(N_{13} - N_7 = 3\right) \Pr\left(N_5 = 4, N_7 - N_5 = 1\right)$$

(since $N_{13} - N_7$ is independent of $N_1, N_2, \ldots, N_7$)

$$= \Pr\left(N_{13} - N_7 = 3\right) \Pr\left(N_7 - N_5 = 1 \middle| N_5 = 4\right) \Pr\left(N_5 = 4\right)$$

$$= \Pr\left(N_{13} - N_7 = 3\right) \Pr\left(N_7 - N_5 = 1\right) \Pr\left(N_5 = 4\right)$$

(since $N_7 - N_5$ is independent of $N_1, N_2, \ldots, N_5$)

$$= \Pr\left(N_6 = 3\right) \Pr\left(N_2 = 1\right) \Pr\left(N_5 = 4\right)$$

$$= \binom{6}{3} p^3 q^3 \binom{2}{1} pq \binom{5}{4} p^4 q = 20 p^8 q^8.$$

To calculate $E\left[N_5 N_8\right]$, we write N_8 as $N_8 = N_5 + \left(N_8 - N_5\right)$. Then

$$E\left[N_5 N_8\right] = E\left[N_5\left(N_5 + \left(N_8 - N_5\right)\right)\right] = E\left[N_5^2 + N_5\left(N_8 - N_5\right)\right]$$

(since $\left(N_8 - N_5\right)$ and N_5 are independent)

$$= E\left[N_5^2\right] + E\left[N_5\right] E\left[N_8 - N_5\right] = E\left[N_5^2\right] + E\left[N_5\right] E\left[N_3\right]$$

$$= \left(5pq + 25p^2\right) + \left(5p\right)\left(3p\right) = 5p\left(q + 8p\right).$$

Since each X_i is independent of X_j, $j \neq i$, we conclude that the process N_n is an independent increment process. The independent increments are <u>stationary</u> because

$$\Pr\left(N_n - N_m = k\right) = \binom{n-m}{k} p^k \left(1-p\right)^{(n-m)^{-k}}, \quad n > m$$

is dependent only on the count difference $(n-m)$ and not on individual counts n and m.

Times of which the Successes of a Bernoulli Process Occur

Denote the times corresponding to the successes in the Bernoulli process by $T_1, T_2, T_3, \cdots$ for example if $X_1 = 0, X_2 = 1, X_3 = 0, X_4 = 1, X_5 = 1, \cdots$ then $T_1 = 2, T_2 = 4, T_3 = 5, \cdots$.

Relations Between Times and Numbers of Successes

Assume that the success number k has occurred at or before the trial number n, this means that $T_n \leq n$. Then the number of successes in the first n trial should be at least k, meaning that $N_n \geq k$: If $T_k \leq n$ then $N_n \geq k$ and the reverse is true, i.e., if $N_n \geq k$ then $T_k \leq n$.

Assume that $T_n = n$, this achieves the presence $k-1$ of successes in the first of the $n-1$ trails and the success of an event in the trial number n, meaning that $N_n = k-1$ and $X_n = 1$. Conversely, if $N_{n-1} = k-1$ and $X_n = 1$ then $T_n = n$.

We will place the previous two relationships as a corollary, and use them to infer the probability distribution of the time T_n with the knowledge of the probability distribution of N_n.

<u>Corollary (1).</u> For integer numbers $k = 1, 2, \ldots$ and $n \geq k$, we have

$$T_k \leq n \text{ iff } N_n \geq k$$

$$T_n = n \text{ iff } N_{n-1} = k-1 \text{ and } X_n = 1$$

<u>Lemma (3)</u>. Let T_n be the time of the n^{th} success in a Bernoulli process $\{X_n, n = 0, 1, \ldots\}$. The sequence of random variables $\{T_n, n = 0, 1, \ldots\}$ is a MC, with transition probabilities

$$p_{ij} = \Pr\left(T_n = j \middle| T_{n-1} = i\right) = \Pr\left(T_n - T_{n-1} = j - i\right) = \begin{cases} pq^{j-i-1}, & j \geq i+1 \\ 0, & \text{otherwise} \end{cases}$$

Here the state space is $SS = \{0, 1, 2, \ldots\}$, $T_0 = 0$, and the TPM of $\{T_n, n = 0, 1, 2, \ldots\}$ is

$$\mathbf{M} = \left(p_{ij}\right)_{i,j} = \begin{bmatrix} 0 & p & pq & pq^2 & pq^3 & \cdots \\ 0 & 0 & p & pq & pq^2 & \cdots \\ 0 & 0 & 0 & p & pq & \cdots \\ \vdots & & & 0 & p & \cdots \\ 0 & & & & \ddots & \ddots \\ & & & & & \end{bmatrix}.$$

The <u>state probability</u> at time t is given by

$$p_n^{(t)} = \Pr\left(T_n = t\right) = \binom{t-1}{n-1} p^n q^{t-n}, \text{ for } t = n, n+1, \ldots$$

with the initial distribution $p_0^{(0)} = 1$; and $p_1^{(0)} = p_2^{(0)} = \ldots = 0$.

The *n*-step transition probabilities of the $\{T_n, n = 0, 1, \ldots\}$ MC can be computed as $p_{ij}^{(n)} = \Pr\left(T_{k+n} = j \middle| T_k = i\right) = \Pr\left(T_{k+n} - T_k = j - i\right)$

$$= \binom{j-i-1}{n-1} p^n q^{j-i-n}, \quad j \geq i+n.$$

The n-step TPM of the $\{T_n, n = 0,1,\ldots\}$ MC is

$$\mathbf{M}^{(n)} = \left(p_{ij}^{(n)}\right)_{i,j} = \begin{bmatrix} 0 & \ldots & 0 & p^n & np^n q & \binom{n+1}{n-1} p^n q^2 & \binom{n+2}{n-1} p^n q^3 & \cdots \\ 0 & & & 0 & p^n & np^n q & \binom{n+1}{n-1} p^n q^2 & \ddots \\ \vdots & & & & \ddots & p^n & \ddots & \\ & & & & & & \ddots & \end{bmatrix}.$$

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