

⋔⊾

Example  $(1)$ . Using the first step-decomposition theorem  $(1)$  given in the previous Lecture # 7. Let  $\{X_n, n=0,1,...\}$  be a MC with TPM  $\mathbf{M} = (p_{ij}),$  on the state space  $SS = \{0,1\}$ . we can easily establish the following:  $=(p_{ij})$ , on the state space  $SS = \{0,1\}$ . we callowing:<br> $f_{00}^{(1)} = Pr(T_{00} = 1) = Pr(X_1 = 0 | X_0 = 0) = p_{00}$ ,

 $f_{00}^{(1)} = Pr(T_{00} = 1) = Pr(X_1 = 0 | X_0 = 0)$ ,  $(T_{00} = n)$  $\{0\}$  $f_{00}^{(1)} = Pr(T_{00} = 1) = Pr(X_1 = 0 | X_0 = 0) = p_{00},$ <br>  $f_{00}^{(n)} = Pr(T_{00} = n) = \sum_{k=0}^{n} p_{0k} f_{k0}^{(n-1)} = p_{01} f_{10}^{(n-1)} = p_{01} p_{11} f_{10}^{(n-2)}$  $\begin{split} f_{00}^{(0)} = \text{Pr}\bigl(T_{00} = 1\bigr) = \text{Pr}\bigl(X_{1} = 0\bigl|X_{0} = 0\bigr) = p_{00}\,,\ f_{00}^{(n)} = \text{Pr}\bigl(T_{00} = n\bigr) = \sum_{k \in SS - \{0\}} p_{0k} f_{k0}^{(n-1)} = p_{01} f_{10}^{(n-1)} = p_{01} p_{11} f_{10}^{(n-1)}\,. \end{split}$  $\begin{split} &f_{00}^{(1)} = \Pr \bigl( \textit{\textbf{T}}_{00} = 1 \bigr) = \Pr \bigl( \textit{\textbf{X}}_{1} = 0 \bigl| \textit{\textbf{X}}_{0} = 0 \bigr) = \textit{\textbf{p}}_{00}, \ &\text{and} \$  $\frac{1}{k}$ <br>*k*  $f_k$ Sollowing:<br>  $f_{00}^{(1)} = Pr(T_{00} = 1) = Pr(X_1 = 0 | X_0 = 0) = p_{00},$ <br>  $f_{00}^{(n)} = Pr(T_{00} = n) = \sum_{k \in SS - \{0\}} p_{0k} f_{k0}^{(n-1)} = p_{01} f_{10}^{(n-1)} = p_{01} p_{11} f_{10}^{(n-2)}$  $= ... = p_{01} (p_{11})^{n-2}$  $\sum_{k \in SS - \{0\}}^{n} p_{01}(p_{11})^{n-2} p_{10}, n \ge 2$ −  $= ... = p_{01}(p_{11})^{n-2} p_{10}, n \ge 2,$ Similarity,  $({\,{}_{{P}_{00}}\,})'$  $P_{11}^{(n)} = \begin{cases} p_{11}, \\ p_{10} ( p_{00} )^{n-2} p_{01}. \end{cases}$  $p_{01}(p_{11})^{n-2} p_{10}, n$ <br>,  $n=1$  $n = 1$ <br> $n \ge 2$ *n n*  $p_{01} = p_{01} (p_{11})^{n-2} p_1$ <br> $p_{11}$ , n *f*  $p_{11}$ , n<br>  $p_{10} (p_{00})^{n-2} p_{01}$ , n −  $\int p_{11}$ ,  $n=$  $=\left\{$  $\left( p_{10} (p_{00})^{n-2} p_{01}, n \geq 0 \right)$ ,  $f_{01}^{(n)} = (p_{00})^{n-1}$  $P_{01}^{(n)} = (p_{00})$   $p_{01}$  $f_{01}^{(n)} = (p_{00})^{n-1} p_{01}$ , and  $f_{10}^{(n)} = (p_{11})^n$  $(n) - (n)^{n-1}$  $p_{10}^{(n)} = (p_{11})$   $p_{10}$  $f_{10}^{(n)} = (p_{11})^{n-1} p_{10}$ , for  $n \ge 1$ .

## **Bernoulli Trials**:

 Consider the tossing of an unfair coin with success given by {head =  $s$ } with probability p and failure given by {tail =  $f$ } with probability q, and  $p+q=1$ . The sample space is  $S_1 = \{s, f\}$ . If the coin is tossed <u>twice</u>, then the sample space  $S_2 = \{ss, sf, fs, ff\}$  is an ordered set from the Cartesian product  $S_1 \times S_1$ . The cardinality of  $S_2$  is  $2^2 = 4$ . The probability of two successes is  $p^2$  and two failures is  $q^2$ . If we toss the coin *n* times, then the resulting sample space is  $S_n = S_1 \times S_1 \times ... \times S_1$  and the cardinality of  $S_n$  is  $2^n$ . The probability of *n* successes is  $p^n$ .

# **Def**. (Bernoulli trials)

Bernoulli trials are repeated functionally independent trials with only two events  $s$  and  $f$  for each trial. The trials are also statistically independent with the two events *s* and *f* in each trial having probabilities p and  $q=1-p$ , respectively. These are called independent identically distributed (iid) trials.

**Def**. (Bernoulli process)

If  $X_n$  is a random variable denotes the number of successes in the trial *n*, The stochastic process  $\{X_n, n=1,2,...\}$  is called Bernoulli process with probability of success  $p(0 \le p \le 1)$ , if it satisfies:

1- The random variables  $X_1, X_2, \cdots$  are independent

2- The event  $\{X_{n} = 1\}$  denotes to the fail in the trial number *n*, while the event  $\{X_{n}=0\}$  denotes to the success in the trial *n*, i.e., denotes to the success in<br>
1 for success s in the  $n<sup>th</sup>$  trial

$$
X_n = \begin{cases} 1 & \text{for success } s \text{ in the } n^{\text{th}} \text{trial} \\ 0 & \text{for failure } f \text{ in the } n^{\text{th}} \text{trial} \end{cases}
$$

with probabilities  $Pr(X_n = 1) = p$ , and  $Pr(X_n = 0) = q = 1 - p \forall n = 1, 2$ , In the Bernoulli process the parameter set is  $T = \{1, 2, ...\}$  and the state space is  $SS = \{0,1\}$  the two are discrete.

The statistics of Bernoulli process Mean: ace is  $SS = \{0,1\}$  the two are discrete.<br>
tistics of Bernoulli process<br>  $E[X_n] = \mu_{X_n} = 1 \cdot Pr(X_n = 1) + 0 \cdot Pr(X_n = 0) = p$ Second moment:  $E|X_n^2| = \mu_{V^2} = 1^2 \cdot Pr(X_n = 1) + 0^2 \cdot Pr(X_n = 0)$ **Pr**( $X_n = 1$ ) + 0.Pr( $X_n = 0$ ) = *p*<br>  $\mu_{X_n^2} = 1^2$ .Pr( $X_n = 1$ ) + 0<sup>2</sup>.Pr( $X_n = 0$ E  $\frac{\text{Bernoulli process}}{P_n} = 1. \Pr(X_n = 1) + 0. \Pr(X_n = 0) = p$ <br> $E[X_n^2] = \mu_{X_n^2} = 1^2. \Pr(X_n = 1) + 0^2. \Pr(X_n = 0) = p$ Variance:  $Var(X_n) = E[X_n^2] - (E[X_n])^2 = p - p^2 = p(1-p)$  $Z_{X_n} = 1 \cdot \Pr(X_n = 1) + 0 \cdot \Pr(X_n = 0) = p$ <br>  $E\left[X_n^2\right] = \mu_{X_n^2} = 1^2 \cdot \Pr(X_n = 1) + 0^2 \cdot \Pr(X_n = 0) = p$ <br>  $Var(X_n) = E\left[X_n^2\right] - \left(E[X_n]\right)^2 = p - p^2 = p(1-p)$ Probability generating function  $(z) = E | z^{x_n} | = z^0 \times Pr(X_n = 0) + z^1 \times Pr(X_n = 1)$ <u>n  $\frac{1}{2}$   $\frac{1}{2}$ 

Autocorrelation function:

$$
\begin{aligned}\n\text{cocorrelation function:} \\
R_X(m,n) &= E[X_m X_n] = \begin{cases} 1^2 \cdot p + 0^2 \cdot (1-p) = p, \ m = n \\ E[X_m] E[X_n] = p^2, \ m \neq n \end{cases} \\
\text{co-covariance:} \\
ov(X_m, X_n) &= R_X(m,n) - E[X_m] E[X_n] = \begin{cases} p - p^2 = p(1-p), \ m = n \\ n^2 - n^2 = 0, \ m \neq n \end{cases}\n\end{aligned}
$$

Auto-covariance:

$$
R_X(m,n) = E[X_m X_n] = \begin{cases} 1^2 \cdot p + 0^2 \cdot (1-p) = p, & m = n \\ E[X_m]E[X_n] = p^2, & m \neq n \end{cases}
$$
\nAuto-covariance:

\n
$$
Cov(X_m, X_n) = R_X(m,n) - E[X_m]E[X_n] = \begin{cases} p - p^2 = p(1-p), & m = n \\ p^2 - p^2 = 0, & m \neq n \end{cases}
$$
\nNormalized auto-covariance:

\n
$$
\rho_X(m,n) = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}
$$

Since the mean is independent of time *n* and the autocorrelation depends only on the time difference  $m - n$ , the process  $X_n$  is stationary.

# **Def**. (Binomial process)

If  $\{X_n, n=1,2,...\}$  represents Bernoulli process with probability of success,  $p(0 \le p \le 1)$ , and  $N_n$  represents the number of successes during the first trials until the completion of the trial *n*:<br> $N = \begin{cases} X_1 + X_2 + \cdots + X_n, n \ge 1 \end{cases}$ 

$$
N_n = \begin{cases} X_1 + X_2 + \dots + X_n, \, n \ge 1 \\ 0, \, n = 0 \end{cases}
$$

where the increments  $\{X_i\}$  form a family of independent  $\{0,1\}$ -

valued random variables.

The process  $\{N_n, n=1,2,...\}$  is called a <u>binomial</u> process.

# The statistics of the binomial process

Mean: Since the probability of each success is  $p$ , the mean value of  $N_n$  can be obtained using the linearity of the expectation operator:

$$
\mu_{N_n} = E[N_n] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = nE[X_i] = np
$$
  
amount: The second moment of N, can be given by

Second moment: The second moment of  $N_n$  can be given by

$$
\mu_{N_n} = E[N_n] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = nE[X_i] = np
$$
  
Second moment: The second moment of  $N_n$  can be given by  

$$
E[N_n^2] = E\left[\left(\sum_{i=1}^n X_i\right)^2\right] = \sum_{i=1}^n E[X_i^2] + \sum_{i \neq j} \sum_{j} E[X_i] E[X_j] = np + n(n-1)p^2
$$
  
Variance:  

$$
Var(N_n) = E[N_n^2] - (E[N_n])^2 = np + n(n-1)p^2 - n^2p^2 = np(1-p).
$$
  
Probability generating function

### Variance:

Variance:  
\n
$$
Var(N_n) = E[N_n^2] - (E[N_n])^2 = np + n(n-1)p^2 - n^2p^2 = np(1-p).
$$
\nProbability generating function  
\n
$$
G_{N_n}(z) = E[z^{N_n}] = E[z^{X_1 + X_2 + ... X_n}] = E[z^{X_1}]E[z^{X_2}]...E[z^{X_n}] = (q + zp)^n
$$
\nAutocorrelation:

Probability generating function<br> $G_{N_n}(z) = E[z^{N_n}] = E[z^{x_1 + x_2 + ... x_n}] = E[z^{x_1}]E[z^{x_2}]...E[z^{x_n}] = (q + zp)^n$  $\begin{aligned} \mathcal{L}^{m} &\bigg[ E\big[ Z^{-1} \big] E\big[ Z^{-1} \big] ... E \big] \ &\bigg[ \sum_{i=1}^{n} \sum_{i=1}^{m} X_{i} X_{i} \big] \end{aligned}$ 

$$
G_{_{N_n}}(z) = E[z^{_{N_n}}] = E[z^{_{X_1+X_2+\ldots X_n}}] = E[z^{_{X_1}}]E[z^{_{X_2}}]...E[z^{_{X_n}}] = (q+zp)^n
$$
  
\nAutocorrelation:  
\n
$$
R_N(m,n) = E[N_mN_n] = E\left[\sum_{j=1}^n \sum_{i=1}^m X_i X_j\right]
$$
\n
$$
= E\left[\sum_{i=1}^m X_i^2\right] + \sum_{j=1}^m \sum_{\substack{i=1 \ i \neq j}}^m E[X_i X_j] + \sum_{j=1}^{n-m} \sum_{\substack{i=1 \ i \neq j}}^m E[X_i X_j]
$$
\n
$$
= mp + m(m-1)p^2 + m(n-m)p^2 = \begin{cases} mp(1-p) + mnp^2 & \text{for } m \le n \\ np(1-p) + mnp^2 & \text{for } n \le m \end{cases}
$$
\n
$$
= p(1-p) \min(m,n) + mnp^2, \ m,n > 0.
$$
  
\nAuto-covariance:  
\n
$$
Cov(N_m, N_n) = R_N(m,n) - E[N_m]E[N_n] = p(1-p) \min(m,n), \ m,n > 0.
$$
  
\nBinomial process is a Markov chain

Auto-covariance:

$$
Cov(N_m, N_n) = R_N(m, n) - E[N_m]E[N_n] = p(1-p)\min(m, n), m, n > 0.
$$
  
**Binomial process is a Markov chain**

If the random variable  $N_n$  denotes the number of successes during the number *n* of Bernoulli's trials, where the probability of success in any one trial is p, the sequence of  $\{N_n, n=1,2,...\}$  is a MC, the probability of transition in one step is

$$
p_{ij} = Pr(N_{n+1} = j | N_n = i) = Pr(X_{n+1} = j - i) = \begin{cases} p, & j = i + 1, \\ 1 - p, & j = i, \\ 0, & \text{otherwise} \end{cases}
$$

and the transition probabilities after *n*-step is given by\n
$$
p_{ij}^{(n)} = \binom{n}{j-i} p^{j-i} q^{n-j+1}, \ j = i, \dots, n+i
$$

If  $N_n$  is the number of successes in the first *n* Bernoulli trials, with probability of a success in any one trial is  $p$ , then the sequence of random variables  $\{N_n, n=1,2,...\}$  is a MC, with  $N_0 = 0$ , since the process  $\{N_n\}$  has the discrete parameter set  $T = \{1, 2, ...\}$ and the discrete state space  $SS = \{0,1,2,...\}$ , and satisfies the Markov property: for all  $i, k, i_1, ..., i_{n-1} \in SS$  we have

Markov property: for all 
$$
i, k, i_1, ..., i_{n-1} \in SS
$$
 we have  
\n
$$
Pr(N_{n+1} = k | N_n = i, N_{n-1} = i_{n-1}, ..., N_1 = i_1)
$$
\n
$$
= \frac{Pr(N_{n+1} = k, N_n = i, N_{n-1} = i_{n-1}, ..., N_1 = i_1)}{Pr(N_n = i, N_{n-1} = i_{n-1}, ..., N_1 = i_1)}
$$
\n
$$
= \frac{Pr(N_{n+1} - N_n = k - i, N_n - N_{n-1} = i - i_{n-1}, ..., N_2 - N_1 = i_2 - i_1, N_1 = i_1)}{Pr(N_n - N_{n-1} = i - i_{n-1}, ..., N_2 - N_1 = i_2 - i_1, N_1 = i_1)}
$$
\n
$$
= \frac{Pr(X_{n+1} = k - i, X_n = i - i_{n-1}, ..., X_1 = i_2 - i_1, N_1 = i_1)}{Pr(X_n = i - i_{n-1}, X_{n-1} = i_{n-1} - i_{n-2}, ..., X_1 = i_2 - i_1, N_1 = i_1)}
$$
\n
$$
= Pr(X_{n+1} = k - i | X_n = i - i_{n-1}, ..., X_1 = i_2 - i_1, N_1 = i_1)
$$
\n
$$
= Pr(X_{n+1} = k - i | N_n = i, N_{n-1} = i_{n-1}, ..., N_1 = i_1)
$$
\n
$$
= Pr(X_{n+1} = k - i | N_n = i, N_{n-1} = i_{n-1}, ..., N_1 = i_1)
$$
\n
$$
= Pr(X_{n+1} = k - i | \sum_{i=1}^{n} X_i = i, \sum_{i=1}^{n-1} X_i = i_{n-1}, ..., N_1 = i_1)
$$

$$
= \Pr(X_{n+1} = k - i) = \Pr(X_{n+1} = k - i) \Pr\left(\sum_{i=1}^{n} X_i = i\right) / \Pr\left(\sum_{i=1}^{n} X_i = i\right)
$$
  
=  $\Pr(X_{n+1} = k - i, \sum_{i=1}^{n} X_i = i) / \Pr\left(\sum_{i=1}^{n} X_i = i\right) = \Pr(X_{n+1} = k - i | \sum_{i=1}^{n} X_i = i\right)$   
=  $\Pr(X_{n+1} = k - i | N_n = i) = \Pr(N_{n+1} = k | N_n = i),$ 

where the third equality is due to the independence of  $X_{n+1}$  and the other *n* random variables. Thus, the future of a process depends only on the most recent past outcome. variables. Thus, the future of a process de<br>t recent past outcome.<br> $Pr(N_{n+1} = k | N_n = i, N_{n-1} = i_{n-1},..., N_1 = i_1)$  depe

The probability depends on the value of  $N_n$  and is independent of the values of  $N_1, N_2, ..., N_{n-1}$ , since  $N_{n+1} = N_n + X_{n+1}$ , i.e.,

$$
\Pr(N_{n+1} = j | N_n = i) = p, \quad \Pr(N_{n+1} = i | N_n = i) = 1 - p.
$$

So,  $N_{n+1}$  depends only on  $N_n$ , and both the state space and the parameter set are discrete, then the process  $\{N_n, n=1,2,...\}$  is an example of MC, on the state space  $SS = \{0,1,2,...\}$ , with parameter set  $T = \{1, 2, ...\}$  and one-step transition probability: ss {*N<sub>n</sub>*, *n* = 1, 2, ...} is an<br>0,1,2, ...}, with parameter<br>bability:<br> $\begin{cases} p, & j = i + 1, \\ 1 - p, & j = i, \end{cases}$ 

example of MC, on the state space 
$$
SS = \{0,1,2,...\}
$$
, with parameter  
set  $T = \{1,2,...\}$  and one-step transition probability:  
 $p_{ij} = Pr(N_{n+1} = j | N_n = i) = Pr(X_{n+1} = j - i) = \begin{cases} p, & j = i+1, \\ 1-p, & j = i, \\ 0, & \text{otherwise} \end{cases}$   
The TPM is  $\mathbf{M} = (p_{ij})_{i,j \in SS} = 2 \begin{bmatrix} 1 & 0 & 1-p & p & 0 & \cdots \\ 0 & 0 & 1-p & p & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$ .

In addition, the MC  ${N_n}_{n \in \mathbb{N}}$  is <u>time homogeneous</u> if the random sequence  ${X_n}_{n \geq 1}$  is identically distributed.

# **Lemma** (1) (Probability distribution)

The number  $N_n$  of successes in the first *n* Bernoulli trials, with probability of a success in any one trial is  $p$ , is a **binomial** random variable: probability of a success in any one trial is  $p$ ,<br>variable:<br> $Pr(k$  successes in any sequence in *n* trials)

= Pr(N<sub>n</sub> = k) = 
$$
\binom{n}{k} p^k q^{n-k}
$$
, for  $p+q=1$ ,  $k = 0,1,...,n$ . (i)

**Proof.** The state probability at time  $n+1$  can be determined from the relation:  $p_k^{(n+1)} = \sum p_i^{(n)}$  $p_k^{(n+1)} = \sum p_i^{(n)} p_{ik}$ *i*  $p_k^{(n+1)} = \sum p_i^{(n)} p_k$  $p_i^{(n)} = \sum p_i^{(n)} p_{ik}$ , i.e., e relation:  $p_k^{(n+1)} = \sum_i p_i^{(n)} p_{ik}$ , i.e.,<br>  $p_k^{(n+1)} = \Pr(N_{n+1} = k) = \sum_i \Pr(N_n = i) \Pr(N_{n+1} = k | N_n = i),$ 

$$
p_k^{(n+1)} = \Pr(N_{n+1} = k) = \sum_i \Pr(N_n = i) \Pr(N_{n+1} = k | N_n = i),
$$

since  $(N_{n+1} = k | N_n = i) = Pr(X_{n+1} = N_{n+1} - N_n = k - i)$ ,  $P = Pr(N_{n+1} = k) = \sum_{i} Pr(N_n = i) Pr(N_{n+1} = k | N_n = i),$ <br>  $Pr(N_{n+1} = k | N_n = i) = Pr(X_{n+1} = N_{n+1} - N_n = k - i) = \begin{cases} q, & i = k \\ p, & i = k - 1 \\ 0, & \text{otherwise} \end{cases}$  $q, i = k$ <br>p,  $i = k - 1$ <br>0, otherwise  $I_{n+1} = k | N_n = i$   $= Pr(X_{n+1} = N_{n+1} - N_n)$ *i*),<br>*q*, *i* = *k*<br>*p*, *i* = *k*  $Pr(N_{n+1} = k) = \sum_{i} Pr(N_{n} = i) Pr(N_{n+1} = k | N_{n} = i),$ <br>  $N_{n+1} = k | N_{n} = i) = Pr(X_{n+1} = N_{n+1} - N_{n} = k - i) = \begin{cases} q, & i = k \\ p, & i = k \\ 0, & \text{other} \end{cases}$  $\int q, i =$  $\vert$  $\begin{aligned} N_{n+1} &= k \big) = \sum_{i} \Pr(N_n = i) \Pr(N_{n+1} = k \big| N_n = i \big), \\ &= k \big| N_n = i \big) = \Pr(X_{n+1} = N_{n+1} - N_n = k - i) = \begin{cases} q, & i = k \\ p, & i = k - 1 \end{cases} \end{aligned}$  $\lfloor$ ,

the following recurrence relation follows

the following recurrence relation follows  
\n
$$
p_k^{(n+1)} = \Pr(N_{n+1} = k) = p \Pr(N_n = k - 1) + q \Pr(N_n = k).
$$
 (ii)

Using the mathematical induction with the recurrence relation (ii), we will prove formula (i):

For  $n = 0$ , formula (i) is true, since  $N_0 = 0$ .

Assume that it is also true for  $n = m$  and all k. That is

$$
\text{Pr}(N_m = k) = \binom{m}{k} p^k q^{m-k}, \text{ for } k = 0, 1, \dots, m. \quad \text{(iii)}
$$

We will prove it is true for  $n = m+1$ , i.e.,<br> $Pr(N = k) = {m+1 \choose m} p^k a^{m+1-k}$ 

Pr(
$$
N_{m+1} = k
$$
) =  $\binom{m+1}{k} p^k q^{m+1-k}$ . (iv)

Putting formula (iii) in the recurrence relation (ii), we get formula (iii) in the recurrence relation (ii), we get<br>Pr  $(N_{m+1} = k) = p \Pr(N_m = k - 1) + q \Pr(N_m = k)$ 

L.H.S. 
$$
Pr(N_{m+1} = k) = p Pr(N_m = k - 1) + q Pr(N_m = k)
$$
  
\n
$$
= p {m \choose k-1} p^{k-1} q^{m-k+1} + q {m \choose k} p^k q^{m-k}
$$
\n
$$
= {m \choose k-1} p^k q^{m-k+1} + {m \choose k} p^k q^{m-k+1} = \left[ {m \choose k-1} + {m \choose k} \right] p^k q^{m-k+1}
$$
\n
$$
= {m+1 \choose k} p^k q^{m-k+1} = R.H.S., \text{ for } 0 < k \le m+1
$$
\nwhen  $k = 0$ , we get  $Pr(N_{m+1} = 0) = p \times 0 + q Pr(N_m = 0) = q \times q^m = q^{m+1}$ 

when  $k = 0$ , we get  $Pr(N_{m+1} = 0) = p \times 0 + q Pr(N_m = 0) = q \times q^m = q^{m+1}$ + Therefore, the probability of k successes in *n* trials is given by a binomial distribution:  $Pr(N_n = k) = {n \choose k} p^k q^{n-k}$ *n*  $N_n = k$  =  $\binom{n}{k} p^k q^k$ cesses in  $\binom{n}{r}$ <sub>n<sup>k</sup>a<sup>n-</sup></sub>  $\kappa$  successes in<br>=  $k$ ) =  $\binom{n}{k} p^k q^{n-k}$ , for  $k = 0, 1, \ldots, n$ .

binomial distribution: 
$$
Pr(N_n = k) = {n \choose k} p^k q^{n-k}
$$
, for  $k = 0, 1, ..., n$ .  
\n**The**  $n^{th}$ **-step transition probabilities**  
\n
$$
p_{ij}^{(n)} = Pr(N_{m+n} = j | N_m = i) = Pr\left(\sum_{k=1}^{m+n} X_k = j | \sum_{k=1}^{m} X_k = i\right)
$$
\n
$$
= Pr\left(\sum_{k=m+1}^{m+n} X_k = j - i\right) = Pr(N_n = j - i)
$$

*n*

$$
= \begin{cases} {n \choose j-i} p^{j-i} q^{n-(j-i)}, \ j = i, i+1, \dots, i+n; \ p+q = 1 \\ 0, \qquad \qquad j < i \end{cases}
$$

**The** *th n* -**step TPM**

$$
p^{th}\text{-step TPM}
$$
\n
$$
\mathbf{M}^{(n)} = (p_{ij}^{(n)})_{i,j \in S^S} = \begin{bmatrix}\n q^n & npq^{n-1} & n \choose 2 & p^2q^{n-2} & n \choose 3 & p^3q^{n-3} & \cdots & n \choose j & p^jq^{n-j} & \cdots \\
 0 & q^n & npq^{n-1} & nq^{n-1} & nq^{n-2} & \cdots & n \choose j-1 & p^{j-1}q^{n-(j-1)} & \cdots \\
 0 & 0 & q^n & npq^{n-1} & \cdots & \cdots & \vdots \\
 0 & 0 & q^n & \cdots & \cdots & \cdots & \vdots \\
 0 & 0 & \cdots & \cdots & \cdots & 0 & \cdots\n\end{bmatrix}.
$$

 Unlike the Bernoulli process, the mean of the binomial process is dependent on time *n* and the autocorrelation function is dependent on both *m* , *n* and hence is a **non-stationary** process.

Note that the increment  $N_{n+m} - N_m$  represents the number of rement  $N_{n+m} - N_m$  1<br>
the trials  $m+1, m$ <br>  $\sum_{m+n}^{m+n}$   $\sum_{m}^{m}$   $\sum_{m}^{m}$   $\sum_{m}^{m+n}$ ement  $N_{n+m} - N_m$  represe<br>
a the trials  $m+1, m+2, ...$ <br>  $\sum_{n=m}^{m} N_n = N_m$ 

is dependent on time *n* and the autocorrelation function is  
dependent on both *m*, *n* and hence is a **non-stationary** process.  
Note that the increment 
$$
N_{n+m} - N_m
$$
 represents the number of  
successes through the trials  $m+1, m+2, ..., m+n$ :  

$$
N_{m+n} - N_m = \sum_{k=1}^{m+n} X_k - \sum_{k=1}^{m} X_k = \sum_{k=m+1}^{m+n} X_k = X_{m+1} + X_{m+2} + \dots + X_{m+n}
$$

It is also a sum of *n* independent random variables that have the same distribution of the Bernoulli distribution, from which we conclude that also a sum of *n* independent random variables that have<br>e distribution of the Bernoulli distribution, from which w<br>clude that<br> $Pr(N_{m+n} - N_m = j) = Pr(N_n = j) = {n \choose j} p^j (1-p)^{n-j}, j = 0,1,...,$ ution, irc $\int^j (1-p)^{n-j}$ o a sum of *n* independent random variables that have the stribution of the Bernoulli distribution, from which we de that  $N_{m+n} - N_m = j$  =  $Pr(N_n = j) = {n \choose j} p^j (1-p)^{n-j}$ ,  $j = 0,1,...,n$ ndom variables th<br>listribution, from<br> $\binom{n}{i} p^j (1-p)^{n-j}$ , um of *n* independent random variables that have the<br>ution of the Bernoulli distribution, from which we<br>at<br> $-N_m = j$ ) = Pr( $N_n = j$ ) =  $\binom{n}{j} p^j (1-p)^{n-j}$ ,  $j = 0,1,...,n$ .

clude that  
Pr
$$
(N_{m+n} - N_m = j)
$$
 = Pr $(N_n = j)$  =  ${n \choose j} p^j (1-p)^{n-j}$ ,  $j = 0,1,...,n$ .

It is the element number *j* in the binomial expansion  $(p+q)^n$  and it does not depend on *m* where  $q = 1-p$ , for example

$$
Pr(N_6 = 3) = {6 \choose 3} p^3 (1-p)^{6-3} = 20 p^3 (1-p)^3
$$
  
Pr(N<sub>15</sub> - N<sub>10</sub> = 4) = Pr(N<sub>5</sub> = 4) =  ${5 \choose 4} p^4 (1-p)^{5-4} = 5 p^4 (1-p)$ .  
**Lemma** (2): The conditional probability of the number of

**Lemma** (2): The conditional probability of the number of **Lemma** (2): The conditional probability of the number of successes,  $N_{n+m} - N_m$  during number of trials  $m+1, m+2, ..., m+n$ , is independent of the number of previous successes until trial number<br> *m* "which is  $N_1, N_2, ..., N_m$ ", for  $k = 0, 1, ..., n$ <br>  $Pr(N_{m+n} - N_m = k | N_m = k_m, N_{m-1} = k_{m-1}, ..., N_1 = k_1)$ 

$$
m \text{ "which is } N_1, N_2, \dots, N_m \text{ ", for } k = 0, 1, \dots, n
$$
\n
$$
\Pr\left(N_{m+n} - N_m = k \, \big| N_m = k_m, N_{m-1} = k_{m-1}, \dots, N_1 = k_1\right)
$$
\n
$$
= \Pr\left(N_{m+n} - N_m = k\right) = \Pr\left(N_n = k\right) = \binom{n}{k} p^k \left(1 - p\right)^{n-k}.
$$

Proof. Using the definition of a random variable *Nn* , we find that the random variables  $N_1, N_2, ..., N_m$  are exact determined in terms of  $X_1, X_2, \ldots, X_m$  and vice versa, i.e., are exact determined in terms of  $\{N_1, N_2, ..., N_m\} \Leftrightarrow \{X_1, X_2, ..., X_m\}$ : Tables  $N_1, N_2, ..., N_m$  are exact determined<br>
nd vice versa, i.e.,  $\{N_1, N_2, ..., N_m\} \Leftrightarrow \{X_1, X_2, ..., X_n\}$ <br>  $X_1 = N_1, X_2 = N_2 - N_1, ..., X_m = N_m - N_{m-1}.$  $\cdots$ ,  $\cdots$ .

Thus, ...,  $X_m$  and vice versa, i.e., { $N_1, N_2, ..., N_m$ } ⇔<br>  $X_1 = N_1, X_2 = N_2 - N_1, ..., X_m = N_m - N_m$ <br>
Pr( $N_{m+n} - N_m = k | N_m = k_m, N_{m-1} = k_{m-1}, ..., N_1 = k_1$ ) nd vice versa, i.e.,  $\{N_1, N_2, ..., N_m\} \Leftrightarrow \{X_1, X_2$ <br>  $X_1 = N_1, X_2 = N_2 - N_1, ..., X_m = N_m - N_{m-1}$ .<br>  $-N_m = k | N_m = k_m, N_{m-1} = k_{m-1}, ..., N_1 = k_1\}$  $X_1 = N_1, X_2 = N_2 - N_1, ..., X_m = N_m - N_{m-1}.$ <br>
Thus,  $Pr(N_{m+n} - N_m = k | N_m = k_m, N_{m-1} = k_{m-1}, ..., N_1 = k_1)$ <br>  $= Pr(N_{m+n} - N_m = k | X_m = x_m, X_{m-1} = x_{m-1}, ..., X_1 = x_1),$  $=\Pr(N_{m+n}-N_m=k|X_m=x_m,X_{m-1}=x_{m-1},...,X_1=x_1),$  $(\text{with } x_i = k_i - k_{i-1}, \text{ for } i = 1, 2, \dots)$ Thus,  $Pr(N_{m+n} - N_m = k | N_m = k_m, N_{m-1} = k_{m-1}, ..., N_1 = k_1)$ <br>=  $Pr(N_{m+n} - N_m = k | X_m = x_m, X_{m-1} = x_{m-1}, ..., X_1 = x_1),$ <br>(with  $x_i = k_i - k_{i-1}$ , for  $i = 1, 2, ...$ )<br>=  $Pr(X_{m+1} + X_{m+2} + ... + X_{m+n} = k | X_m = x_m, X_{m-1} = x_{m-1}, ..., X_1 = x_1)$ <br>In other meaning  $N_{m+1} = N_{m+1$ In other meaning  $N_{m+n} - N_m = X_{m+1} + X_{m+2} + ... + X_{m+n}$  and  $X_{m+2} + ... + X_{m+n} = k \Big| X_m = x_m, X_{m-1} = x$ <br>  $N_{m+n} - N_m = X_{m+1} + X_{m+2} + ... + X_{m+n}$  an

In other meaning 
$$
N_{m+n} - N_m = X_{m+1} + X_{m+2} + ... + X_{m+n}
$$
 and  
\n $\{X_{m+1}, X_{m+2},..., X_{m+n}\}$  are independent of  $\{X_m, X_{m-1},..., X_1\}$ . Thus  
\n $Pr(N_{m+n} - N_m = k | X_m = x_m, X_{m-1} = x_{m-1},..., X_1 = x_1)$   
\n $= Pr(N_{m+n} - N_m = k) = Pr(N_n = k) = {n \choose k} p^k (1-p)^{n-k}, k = 0,1,...,n$ 

**Corollary** (1). If  $n_1 > \cdots > n_n > n_1 > n_0 = 0$  are positive integers, then **Corollary** (1). If  $n_j > \cdots > n_2 > n_1 > n_0 = 0$  are positive integers, then the random variables (increments)  $N_{n_j} - N_{n_{j-1}}, ..., N_{n_2} - N_{n_1}, N_{n_1} - N_{n_0}$ are independent.

**Example** (1). Find the following  
- the JPMF Pr(
$$
N_{13} = 8, N_7 = 5, N_5 = 4
$$
)

- the expected value  $E[N_sN_s]$ .

- the expected value 
$$
E[N_sN_s]
$$
.  
\nSolution. Since the following two events are equivalent  
\n
$$
\{N_{13} = 8, N_7 = 5, N_5 = 4\} \text{ and } \{N_5 = 4, N_7 - N_5 = 1, N_{13} - N_7 = 3\},\
$$
\nthen  $Pr(N_{13} = 8, N_7 = 5, N_5 = 4) = Pr(N_5 = 4, N_7 - N_5 = 1, N_{13} - N_7 = 3)$   
\n
$$
= Pr(N_{13} - N_7 = 3|N_5 = 4, N_7 - N_5 = 1)Pr(N_5 = 4, N_7 - N_5 = 1)
$$
\n
$$
= Pr(N_{13} - N_7 = 3)Pr(N_5 = 4, N_7 - N_5 = 1)
$$
\n(since  $N_{13} - N_7$  is independent of  $N_1, N_2, ..., N_7$ )  
\n
$$
= Pr(N_{13} - N_7 = 3)Pr(N_7 - N_5 = 1|N_5 = 4)Pr(N_5 = 4)
$$
\n
$$
= Pr(N_{13} - N_7 = 3)Pr(N_7 - N_5 = 1)Pr(N_5 = 4)
$$
\n(since  $N_7 - N_5$  is independent of  $N_1, N_2, ..., N_5$ )  
\n
$$
= Pr(N_6 = 3)Pr(N_2 = 1)Pr(N_5 = 4)
$$
\n
$$
= {6 \choose 3} p^3 q^3 {2 \choose 1} pq \begin{pmatrix} 5 \\ 4 \end{pmatrix} p^4 q = 20p^8 q^8.
$$
\nTo calculate  $E[N_5N_8]$ , we write  $N_8$  as  $N_8 = N_5 + (N_8 - N_5)$ . Then  
\n
$$
E[N_5N_8] = E[N_5(N_5 + (N_8 - N_5))] = E[N_5^2 + N_5(N_8 - N_5)]
$$
\n(since  $(N_8 - N_5)$  and  $N_5$  are independent)  
\n
$$
= E[N_5^2] + E[N_5]E[N_8 - N_5] = E[N_5^2] + E[N_5]E[N_3]
$$
\n
$$
= (5pq + 25p^2) + (5p)(3p) = 5p(q + 8
$$

Since each  $X_i$  is independent of  $X_j$ ,  $j \neq i$ , we conclude that the process  $N_n$  is an independent increment process. The independent increments are <u>stationary</u> because<br>  $Pr(N_n - N_m = k) = {n-m \choose k} p^k (1-p)^{(n-m)^{-k}}$ ,  $n > m$ increments are stationary because  $\frac{1}{2}$  increase  $n-m$ t increment process<br>
cause<br>  $\binom{n-m}{r} p^k (1-p)^{(n-m)}$ independent increment process. The independent<br>ationary because<br> $-N_m = k$ ) =  $\binom{n-m}{k} p^k (1-p)^{(n-m)^{-k}}$ ,  $n > m$ 

are stationary because  
Pr(N<sub>n</sub> - N<sub>m</sub> = k) = 
$$
\binom{n-m}{k} p^{k} (1-p)^{(n-m)^{-k}}
$$
,  $n > m$ 

is dependent only on the count difference  $(n-m)$  and not on individual counts *n* and *m*.

# Times of which the Successes of a Bernoulli Process Occur

Denote the times corresponding to the successes in the Bernoulli process by  $T_1, T_2, T_3, \cdots$  for example if Denote the times corresponding to the su<br>process by  $T_1$ ,  $T_2$ ,  $T_3$ ,  $\cdots$  for example if<br> $X_1 = 0, X_2 = 1, X_3 = 0, X_4 = 1, X_5 = 1, \cdots$  then

then  $T_1 = 2, T_2 = 4, T_3 = 5, \dots$ .

## Relations Between Times and Numbers of Successes

Assume that the success number *k* has occurred at or before the trial number *n*, this means that  $T_n \leq n$ . Then the number of successes in the first *n* trial should be at least *k* , meaning that  $N_n \ge k$ : If  $T_k \le n$  then  $N_n \ge k$  and the reverse is true, i.e., if  $N_n \ge k$ then  $T_k \leq n$ .

Assume that  $T_n = n$ , this achieves the presence  $k-1$  of successes in the first of the  $n-1$  trails and the success of an event in the trial number *n*, meaning that  $N_n = k - 1$  and  $X_n = 1$ . Conversely, if  $N_{n-1} = k - 1$  and  $X_n = 1$  then  $T_n = n$ .

We will place the previous two relationships as a corollary, and use them to infer the probability distribution of the time  $T_n$  with the knowledge of the probability distribution of  $N<sub>n</sub>$ .

**Corollary** (1). For integer numbers  $k = 1, 2, ...$  and  $n \ge k$ , we have

$$
T_k \le n \quad \text{iff} \quad N_n \ge k
$$
\n
$$
T_n = n \quad \text{iff} \quad N_{n-1} = k - 1 \quad \text{and} \quad X_n = 1
$$

**Lemma** (3). Let  $T_n$  be the time of the  $n^{th}$  success in a Bernoulli<br>
process { $X_n$ ,  $n = 0,1,...$ }. The sequence of random variables<br>
{ $T_n$ ,  $n = 0,1,...$ } is a MC, with transition probabilities<br>  $P_{ij} = Pr(T_n = j | T_{n-1} = i) = Pr(T_n - T_{n$ process { $X_n$ ,  $n = 0,1,...$ }. The sequence of random variables<br>{ $T_n$ ,  $n = 0,1,...$ } is a MC, with transition probabilities<br> $p_{ij} = Pr(T_n = j | T_{n-1} = i) = Pr(T_n - T_{n-1} = j - i) = \begin{cases} pq^{j-i-1}, & j \ge i+1 \\ 0, & \text{otherwise} \end{cases}$  $T_n$ ,  $n = 0, 1, ...$ }. The sequence of random va<br>{ $T_n$ ,  $n = 0, 1, ...$ } is a MC, with transition probabilities<br> $p_{ij} = Pr(T_n = j | T_{n-1} = i) = Pr(T_n - T_{n-1} = j - i) = \begin{cases} pq_0, & \text{if } j \neq j_0, \\ 0, & \text{if } j \neq j_0, \end{cases}$ *j i g*. Let  $T_n$  be the time of the  $n^m$  success in a Berno  $\{X_n, n = 0, 1, ...\}$ . The sequence of random variables  $p, 1, ...\}$  is a MC, with transition probabilities<br> $= Pr(T_n = j | T_{n-1} = i) = Pr(T_n - T_{n-1} = j - i) = \begin{cases} pq^{j-i-1}, & j \ge i \\ 0, & \text{other} \$  $-$ <sup>-i-1</sup>,  $j ≥ i +$ ss in a Bernoulli<br>
in variables<br>
ties<br>  $\int pq^{j-i-1}$ ,  $j \ge i+1$ <br>
0. otherwise

$$
p_{ij} = Pr(T_n = j | T_{n-1} = i) = Pr(T_n - T_{n-1} = j - i) = \begin{cases} pq^{j-i-1}, & j \ge i+1 \\ 0, & \text{otherwise} \end{cases}
$$

Here the state space is  $SS = \{0, 1, 2, ...\}$ ,  $T_0 = 0$ , and the TPM of  ${T_n, n = 0,1,2,...}$  is *p pq pq pq*  $\begin{bmatrix} 0 & p & pq & pq^2 & pq^3 & \cdots \end{bmatrix}$ 

is  
\n
$$
\mathbf{M} = (p_{ij})_{i,j} = \begin{bmatrix}\n0 & p & pq & pq^2 & pq^3 & \cdots \\
0 & 0 & p & pq & pq^2 & \cdots \\
0 & 0 & 0 & p & pq & \cdots \\
\vdots & & & 0 & p & \cdots \\
0 & & & & \ddots & \ddots\n\end{bmatrix}.
$$

The state probability at time *t* is given by  
\n
$$
p_n^{(t)} = \Pr(T_n = t) = \binom{t-1}{n-1} p^n q^{t-n}, \text{ for } t = n, n+1,...
$$

with the initial distribution  $p_0^{(0)}$  $p_0^{(0)} = 1$ ; and  $p_1^{(0)} = p_2^{(0)} = ... = 0$ .

The *n*-step transition probabilities of the  $\{T_n, n = 0, 1, ...\}$  MC can be The *n*-step transition probabilities of the  $\{T_n, n = 0, 1, ...\}$  for computed as  $p_{ij}^{(n)} = Pr(T_{k+n} = j | T_k = i) = Pr(T_{k+n} - T_k = j - i)$ (*n* distribution  $p_0 = 1$ , and  $p_1 = p_2 = ... = 0$ .<br>
insition probabilities of the  $\{T_n, n = 0, 1, ...\}$  MC can  $p_{ij}^{(n)} = Pr(T_{k+n} = j | T_k = i) = Pr(T_{k+n} - T_k = j - i)$ 

$$
= \Pr(T_{k+n} = j | T_k = i) = \Pr(T_{k+n} - T_k)
$$

$$
= \binom{j-i-1}{n-1} p^n q^{j-i-n}, \quad j \ge i+n.
$$

The *n*-step TPM of the  $\{T_n, n = 0, 1, ...\}$  MC is

$$
u\text{-step TPM of the }\{T_n, n = 0, 1, ...\} \text{ MC is}
$$
\n
$$
\mathbf{M}^{(n)} = (p_{ij}^{(n)})_{i,j} = \begin{bmatrix}\n0 & \cdots & 0 & p^n & np^n q & \binom{n+1}{n-1} p^n q^2 & \binom{n+2}{n-1} p^n q^3 & \cdots \\
0 & 0 & p^n & np^n q & \binom{n+1}{n-1} p^n q^2 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
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0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 &
$$