المحاضرات السابعة والثامنة

الفرقة: الثالثة

الشعبة: احصاء وعلوم حاسب

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If $V(x,y)$ is any function, the equation

$$P.I. = \frac{1}{A_0(D-m_1)(D-m_2)\cdots(D-m_n)} V(x,y)$$

can be evaluated by solving n equations of first order

$$\left.\begin{aligned} u_1 &= \frac{1}{D-m_n D'} V(x,y) \\ u_2 &= \frac{1}{D-m_{n-1} D'} u_1 \\ &\cdots\cdots \\ u_n &= \frac{1}{D-m_1 D'} u_{n-1} \end{aligned}\right\} \longrightarrow (1)$$

Eeach of equations (1) is of the form

$$p - mq = g(x,y) \longrightarrow (2)$$

Eq. (2) being of first order has its Lagrange's subsidiary equations

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{du}{g(x,y)} \longrightarrow (3)$$

$$-m\,dx = dy$$

$$m\,dx + dy = 0$$

integrating

$y + mx = a$, a — const $\longrightarrow$ (4)

$du = g(x, y)\,dx$

$du = g(x, a - mx)\,dx$

integrating

$$u = \int g(x, a - mx)\,dx \longrightarrow (5)$$

after integrating 'a' is to be replaced by

$$a = y + mx$$

Example: Solve

$$(D - D')(D + 2D')z = (y+1)e^{x}$$

Solution: A.E. is

$$(m-1)(m+2) = 0$$

$$m = 1, -2$$

$$C.F. = \phi_1(y+x) + \phi_2(y-2x)$$

$$P.I. = \frac{1}{(D-D')(D+2D')}(y+1)e^{x}$$

$$= \frac{1}{D-D'}\left\{\int (a+2x+1)e^{x}\,dx\right\}_{a=y-2x}$$

$$= \frac{1}{D-D'}\left\{(a+2x+1)e^{x} - 2e^{x}\right\}$$

$$= \frac{1}{D-D'}\left\{(a+2x-1)e^{x}\right\}_{a=y-2x}$$

$$= \frac{1}{D-D'}(y-1)e^{x}$$

$$= \int (a-x-1)e^{x}\,dx \quad , a = y+x$$

$$= (a-x-1)e^{x} + e^{x}$$

$$= (a-x)e^{x} \quad , a = y+x$$

$$= y e^{x}$$

$$\therefore Z = \phi_1(y+x) + \phi_2(y-2x) + y e^{x}$$

Non-homogeneous linear equations with constant co-efficients.

If all terms on left-hand side of the equation

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) Z = V(x,y) \longrightarrow (1)$$

are not of same degree, eq. (1) is said to be non-homogeneous equation.

Equation is said to be reducible if the symbolic function $f(D, D')$ can be resolved in to factors each of which is of first degree in D and D' otherwise is called irreducible.

e.g. Equation

$$f(D, D')Z = (D^2 - D'^2 + 2D + 1)Z$$
$$= (D + D' + 1)(D - D' + 1)Z = x^2 + xy$$

is reducible.

while equation

$$f(D, D')Z = (DD' + D'^3)Z$$
$$= D'(D + D'^2)Z = \cos(x + 2y)$$

is irreducible.

Reducible non-homogeneous equations

Let

$$f(D, D') = (a_1 D + b_1 D' + c_1)(a_2 D + b_2 D' + c_2) \dots (a_n D + b_n D' + c_n) \longrightarrow (1)$$

where a's, b's and c's are constants

Then the equation $f(D, D') = 0$ takes the form

$$(a_1 D + b_1 D' + c_1)(a_2 D + b_2 D' + c_2) \dots (a_n D + b_n D' + c_n) = 0 \longrightarrow (2)$$

Any solution of the equation

$$(a_i D + b_i D' + c_i)Z = 0 \longrightarrow (3)$$

is a solution of the eq. (2)

From eq. (3)

$(a_i D + b_i D')z = -c_i z$

The Lagrange's subsidiary equations are

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz}{-c_i z} \longrightarrow (4)$$

$$a_i dy = b_i dx$$

$$b_i dx - a_i dy = 0$$

integrating

$$b_i x - a_i y = \text{const}$$

$$-c_i dx = a_i dz \left(\frac{1}{z}\right)$$

$$\frac{1}{z} dz = -\frac{c_i}{a_i} dx$$

$$\ln z + \ln \text{cost} = -\frac{c_i}{a_i} x$$

$$\ln\left(\frac{z}{\text{const}}\right) = -\frac{c_i}{a_i} x$$

$$z = \text{const}\, e^{-\frac{c_i}{a_i} x} \quad \text{if } a_i \neq 0$$

or

$$b_i \frac{dz}{z} = -c_i dy$$

$$z = \text{const}\, e^{-\frac{c_i}{b_i} y} \quad \text{if } b_i \neq 0$$

Therefore, general solution of eq. (3) is

$$Z = e^{-\frac{c_i}{a_i}x} \phi_i(b_i x - a_i y) \text{ if } a_i \neq 0 \longrightarrow (5)$$

or

$$Z = e^{-\frac{c_i}{b_i}y} \psi_i(b_i x - a_i y) \text{ if } b_i \neq 0 \longrightarrow (6)$$

where ϕ_i and ψ_i are arbitrary functions

Example: Solve $(D^2 - D'^2 - 3D + 3D')Z = 0$

Solution

The equation can be written as

$$(D - D')(D + D' - 3)Z = 0$$

$$C.F. = \phi_1(b_1 x - a_1 y)e^{-\frac{c_1}{a_1}x} + \phi_2(b_2 x - a_2 y)e^{-\frac{c_2}{a_2}x}$$

$a_1 = 1, \; b_1 = -1, \; c_1 = 0$

$a_2 = 1, \; b_2 = 1, \; c_2 = -3$

$$C.F. = \phi_1(-x - y) + \phi_2(x - y)e^{3x}$$

or

$$C.F. = \psi_1(b_1 x - a_1 y)e^{-\frac{c_1}{b_1}y} + \psi_2(b_2 x - a_2 y)e^{-\frac{c_2}{b_2}y}$$

$$C.F. = \psi(-x - y) + \psi_2(x - y)e^{3y}$$

where the general solution of eq. (2) is the sum f the general solutions of the equations of the type (3) corresponding to each factors in (1).

Case of repeated factors :

Let two times repeated factors of ① be

$$(aD+bD'+c)$$

Consider the equation

$$(aD+bD'+c)(aD+bD'+c)z=0 \longrightarrow ⑦$$

Put

$$(aD+bD'+c)z=u$$

Then eq. ⑦ reduces to

$$(aD+bD'+c)u=0 \longrightarrow ⑧$$

As above, general solution of eq. ⑧ is

$$u=e^{-\frac{c}{a}x}\phi(bx-ay) \text{ if } a\neq 0 \longrightarrow ⑨$$

or

$$u=e^{-\frac{c}{b}y}\psi(bx-ay) \text{ if } b\neq 0 \longrightarrow ⑩$$

we have $(aD+bD'+c)z=u$

From eq. ⑨

$$(aD+bD'+c)z=e^{-\frac{c}{a}x}\phi(bx-ay)$$

Subsidiary equations are

$$\frac{dx}{a}=\frac{dy}{b}=\frac{dz}{e^{-\frac{c}{a}x}\phi(bx-ay)-cz}$$

$$b\,dx = a\,dy$$
$$b\,dx - a\,dy = 0$$

integrating

$$bx - ay = \lambda, \quad \lambda - \text{const}$$

$$a\,dz = \left(e^{-\frac{c}{a}x}\varphi(bx-ay) - cz\right)dx$$

$$\frac{dz}{dx} + \frac{c}{a}z = \frac{1}{a}e^{-\frac{c}{a}x}\varphi(bx-ay)$$

$$\frac{dz}{dx} + \frac{c}{a}z = \frac{1}{a}e^{-\frac{c}{a}x}\varphi(\lambda) \longrightarrow (11)$$

The eq. (11) being an ordinary linear equation

$$z = e^{-\int\frac{c}{a}dx}\left[\text{Const} + \int e^{\int\frac{c}{a}dx}\,\frac{1}{a}e^{-\frac{c}{a}x}\varphi(\lambda)\,dx\right]$$

$$z = e^{-\frac{c}{a}x}\left[\text{Const} + \int e^{\frac{c}{a}x}\,\frac{1}{a}e^{-\frac{c}{a}x}\varphi(\lambda)\,dx\right]$$

$$z e^{\frac{c}{a}x} = \text{Const} + \frac{1}{a}\varphi(\lambda)\int dx \qquad \mu = e^{\int p(x)dx}$$

$$z e^{\frac{c}{a}x} = \frac{1}{a}\varphi(\lambda)\,x + \text{Const} \qquad z = \frac{1}{\mu}\left[c + \int \mu g\,dx\right]$$

$$z e^{\frac{c}{a}x} = \frac{1}{a}x\varphi(bx-ay) + \text{Const}$$

$$z = \frac{1}{a}x\varphi(bx-ay)e^{-\frac{c}{a}x} + \text{Const}\,e^{-\frac{c}{a}x}$$

Therefore, general solution is

$$Z = \frac{x}{a} e^{-\frac{c}{a}x} \phi(bx - ay) + \phi_1(bx - ay) e^{-\frac{c}{a}x}$$

$$Z = e^{-\frac{c}{a}x} \{ x\phi_2(bx - ay) + \phi_1(bx - ay) \}$$

Similarly from eq. (10)

$$Z = e^{-\frac{c}{b}y} \{ y\psi_2(bx - ay) + \psi_1(bx - ay \}$$

where ϕ_1 and ϕ_2 are arbitrary functions
ψ_1 and ψ_2 are arbitrary functions.

In general, for r times repeated factor
$(aD + bD' + C)$,

$$Z = e^{-\frac{c}{a}x} \{ \sum_{i=1}^{r} x^{i-1} \phi_i(bx - ay) \} \quad \text{if } a \neq 0$$

$$Z = e^{-\frac{c}{b}y} \{ \sum_{i=1}^{r} y^{i-1} \psi_i(bx - ay) \} \quad \text{if } b \neq 0$$

where $\phi_1, \phi_2, ---, \phi_r$ and $\psi_1, \psi_2, ---, \psi_r$ are arbitrary functions.

Example: solve $(2D - D' + 4)(D + 2D' + 1)^2 Z = 0$

Solution: for the factor $(2D - D' + 4)$

$$a = 2, \; b = -1, \; c = 4$$

the part of C.F. is

$e^{4y}\psi(-x-y)$

For the factor $(D+2D'+1)^2$

$a=1,\ b=2,\ c=1$

Part of C.F. is

$$e^{-x}\{x\phi_2(2x-y)+\phi_1(2x-y)\}$$

$$\therefore C.F. = e^{4y}\psi(x+2y)+e^{-x}\{x\phi_2(2x-y)+\phi_1(2x-y)\}$$

Irreducible non-homogeneous equations:

Consider the equation

$$f(D,D')Z=0 \longrightarrow ①$$

To solve eq. ①, put

$$Z=ce^{ax+by} \longrightarrow ②$$

where a, b and c are consts.

$$D^r Z = ca^r e^{ax+by}$$

$$D'^s Z = cb^s e^{ax+by}$$

$$D^r D'^s = cb^s a^r e^{ax+by}$$

Substituting in eq. ①

$$cf(a,b)e^{ax+by}=0$$

$$c\neq 0,\quad e^{ax+by}\neq 0$$

$$f(a,b)=0 \longrightarrow ③$$

For any chosen value of a (or b) eq. 3 gives one or more values of b (or a). Thus there exist infinitely many pairs of numbers (a_i, b_i) satisfying 3.

Thus

$$Z = \sum_{i=1}^{\infty} C_i e^{a_i x + b_i y} \longrightarrow (4)$$

where $f(a_i, b_i) = 0$.

If $f(D, D') = (D + hD' + k)\, g(D, D') \longrightarrow (5)$

any pair of (a, b) such that $a + hb + k = 0$ satifies 3. Such pair are again infinite number.

$$Z = \sum_{i=1}^{\infty} C_i e^{a_i x + b_i y}$$

$$a = -hb - k$$

$$a = -(hb + k)$$

$$Z = \sum_{i=1}^{\infty} C_i e^{-(hb_i + k)x + b_i y}$$

$$= e^{-kx} \sum_{i=1}^{\infty} C_i e^{-b_i(hx - y)}$$

$$= e^{-kx} \sum_{i=1}^{\infty} C_i e^{b_i(y - hx)} \longrightarrow (6)$$

is a Part of C.F. corresponding to a linear factor $(D+hD'+k)$ in ⑤.

eq. ⑥ is equivalent to

$$e^{-kx}\phi(y-hx)$$

where ϕ is an arbitrary function.

Thus if $f(D,D')$ has no linear factor, ④ will be called general solution of the eq. ①.

If $f(D,D')$ has linear factors, general solution will involve Partly arbitrary functions and Partly arbitrary constants.

Example: Solve $(D^2+D+D')Z=0$

Solution

$$Z=\sum_{i=1}^{\infty} c_i e^{a_i x+b_i y}$$

where $f(a_i,b_i)=0$

$$a_i^2+a_i+b_i=0$$

$$b_i=-a_i(a_i+1)$$

$$Z=\sum_{i=1}^{\infty} c_i e^{a_i x-a_i(a_i+1)y}$$

where a_i and c_i are arbitrary constants.

Example: Solve

$$(D-2D')(D-2D'+1)(D-D'^2)z=0$$

Solution: For the factor $(D-2D')$

Part of C.F. is

$$\phi_1(-2x-y)$$

For the factor $(D-2D'+1)$

$$a=1, \quad b=-2, \quad c=1$$

Part of C.F. is

$$e^{-x}\phi_2(-2x-y)$$

For the factor $(D-D'^2)$

Part of C.F. is $\displaystyle\sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$, where

$$a_i - b_i^2 = 0$$

$$a_i = b_i^2$$

$\therefore$ Part of C.F. becomes

$$\sum_{i=1}^{\infty} c_i e^{b_i(b_i x + y)}$$

$$\therefore \text{C.F.} = \phi_1(y-2x) + e^{-x}\phi_2(y-2x) + \sum_{i=1}^{\infty} c_i e^{b_i(b_i x + y)}$$

where ϕ_1, ϕ_2 are arbitrary functions and

c_i, b_i are arbitrary constants.

Particular integral :

Consider the equation

$$f(D, D')Z = V(x,y) \longrightarrow (1)$$

Here $f(D, D')$ is a non-homogeneous function of D and D'.

$$P.I. = \frac{1}{f(D, D')} V(x,y) \longrightarrow (2)$$

If $V(x,y)$ is of the form e^{ax+by} where a and b are constants, we used the following theorem.

Theorem : If $f(a,b) \neq 0$, then

$$\frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$$

Proof : $D^r e^{ax+by} = a^r e^{ax+by}$

$$D'^s e^{ax+by} = b^s e^{ax+by}$$

$$D^r D'^s e^{ax+by} = a^r b^s e^{ax+by}$$

$$\therefore f(D, D') e^{ax+by} = f(a,b) e^{ax+by}$$

$$\frac{1}{f(a,b)} e^{ax+by} = \frac{1}{f(D, D')} e^{ax+by}$$

or $\frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}$

Example: Solve $(D^2 - D'^2 - 3D + 3D')Z = e^{x-2y}$

Solution:

$$(D^2 - D'^2 + 3D' - 3D)Z = e^{x-2y}$$

is equivalent to

$$(D-D')(D+D'-3)Z = e^{x-2y}$$

For the factor $(D-D')$

$$a=1, \quad b=-1, \quad c=0$$

Part of C.F. is

$$\phi_1(-x-y)$$

For the factor $(D+D'-3)$

$$a=1, \quad b=1, \quad c=-3$$

Part of C.F. is

$$\phi_2(x-y)\, e^{3x}$$

$$C.F. = \phi_1(-x-y) + e^{3x}\phi_2(x-y)$$

$$P.I. = \frac{1}{(D-D')(D+D'-3)} e^{x-2y}$$

$$= \frac{1}{(1+2)(1-2-3)} e^{x-2y}$$

$$= -\frac{1}{12} e^{x-2y}$$

$$Z = \phi_1(-x-y) + e^{3x}\phi_2(x-y) - \frac{1}{12} e^{x-2y}$$

If $V(x,y)$ is of the form
$\sin(ax+by)$ or $\cos(ax+by)$.
Then since

$$D^2 \sin(ax+by) = (-a^2)\sin(ax+by)$$
$$DD' \sin(ax+by) = (-ab)\sin(ax+by)$$
$$D'^2 \sin(ax+by) = (-b^2)\sin(ax+by)$$

$$D^2 \cos(ax+by) = (-a^2)\cos(ax+by)$$
$$DD' \cos(ax+by) = (-ab)\cos(ax+by)$$
$$D'^2 \cos(ax+by) = (-b^2)\cos(ax+by)$$

Then the method to evaluate

$$P.I. = \frac{1}{f(D,D')} V(x,y)$$

is to replace D^2 by $(-a^2)$ & D'^2 by $(-b^2)$ and DD' by $(-ab)$.

Example: Solve
$$(D+D')(D+D'-2)Z = \sin(x+2y)$$

Solution: $C.F. = \phi_1(x-y) + e^{2x}\phi_2(x-y)$

$$P.I. = \frac{1}{(D+D')(D+D'-2)}\sin(x+2y)$$

$$\frac{1}{D^2+2DD'+D'^2-2D-2D'}\sin(x+2y)$$

$$= \frac{1}{(-1)+2(-2)+(-4)-2(D+D')} \sin(x+2y)$$

$$= \frac{-1}{9+2D+2D'} \sin(x+2y)$$

$$= -\frac{(9-2D-2D')}{(9+2D+2D')(9-2D-2D')} \sin(x+2y)$$

$$= \frac{(-9+2D+2D')}{81-4(D^2+2DD'+D'^2)} \sin(x+2y)$$

$$= \frac{(-9+2D+2D')}{81-4(-1-4-4)} \sin(x+2y)$$

$$= \frac{1}{117}(-9+2D+2D') \sin(x+2y)$$

$$= \frac{1}{117}\{-9\sin(x+2y)+2\cos(x+2y)+4\cos(x+2y)\}$$

$$= \frac{1}{117}\{6\cos(x+2y)-9\sin(x+2y)\}$$

$$z = \phi_1(x-y) + e^{2x}\phi_2(x-y) + \frac{1}{117}\{6\cos(x+2y) - 9\sin(x+2y)\}$$

If $v(x,y)$ is of the form $x^m y^n$ where m and n are Positive integers, the equation

$$P.I. = \frac{1}{f(D,D')} V(x,y)$$

can be evaluated by expanding the symbolic function $\frac{1}{f(D,D')}$ in an infinite series of asecconding Powers of D and D'.

Example: Solve $(DD' + D - D' - 1)z = xy$

Solution: $(DD' + D - D' - 1)z = xy$

is equivalent to

$$(D'+1)(D-1)z = xy$$

For the factor $(D'+1)$

$$a=0,\ b=1,\ c=1$$

Part of C.F. is

$$\psi_1(x)\, e^{-y}$$

For the factor $(D-1)$

$$a=1,\ b=0,\ c=-1$$

Part of C.F. is

$$\phi_1(-y)\, e^{x}$$

$$C.F. = e^{-y}\psi_1(x) + e^{x}\phi_1(-y)$$

$$P.I. = \frac{1}{(D'+1)(D-1)}\, xy$$

$$= -(1+D')^{-1}(1-D)^{-1}\, xy$$

$$= -\{1 - D' + D'^2 - ---\}\{1 + D + D^2 + ---\}\, xy$$

$$= -\{1 + D - D' - DD' + ---\}\, xy$$

$$= -\{xy + y - x - 1\}$$

$$\therefore z = e^{-y}\psi_1(x) + e^{x}\phi_1(-y) - (xy + y - x - 1)$$