<u>المحاضرات التاسعة والعاشرة</u>

الفرقة: الثالثة

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المادة : معادلات تفاضلية جزئية (336ر)

If $V(x,y)$ be of the form $e^{ax+by}\phi(x,y)$, where a and b are constants, we used the following theorem.

Theorem: If $\phi(x,y)$ be any function, then

$$\frac{1}{f(D,D')}e^{ax+by}\phi(x,y)=e^{ax+by}\frac{1}{f(D+a,D'+b)}\phi(x,y)$$

Proof: By Leibnitz's theorem for successive differentiation

$$D^r\{e^{ax+by}\phi(x,y)\}=e^{ax+by}\{D^r\phi(x,y)+{}^rC_1\,a\,D^{r-1}\phi(x,y)$$

$$+{}^rC_2\,a^2D^{r-2}\phi(x,y)+\cdots+a^r\phi(x,y)\}$$

$$=e^{ax+by}\{D^r+{}^rC_1\,a\,D^{r-1}+{}^rC_2\,a^2D^{r-2}+\cdots+a^r\}\phi(x,y)$$

$$=e^{ax+by}(D+a)^r\phi(x,y)$$

Similarly

$$D'^s\{e^{ax+by}\phi(x,y)\}=e^{ax+by}(D'+b)^s\phi(x,y)$$

$$D^rD'^s\{e^{ax+by}\phi(x,y)\}=D^r\{e^{ax+by}(D'+b)^s\phi(x,y)\}$$

$$=e^{ax+by}(D+a)^r(D'+b)^s\phi(x,y)$$

$$\therefore f(D,D')\{e^{ax+by}\phi(x,y)\}=e^{ax+by}f(D+a,D'+b)\phi(x,y) \longrightarrow (1)$$

Put $f(D+a, D'+b)\,\phi(x,y) = \psi(x,y)$

$$\phi(x,y) = \frac{1}{f(D+a, D'+b)}\,\psi(x,y)$$

Substituting in eq. ①

$$f(D, D')\left\{e^{ax+by}\,\frac{1}{f(D+a, D'+b)}\,\psi(x,y)\right\}$$

$$= e^{ax+by}\,\psi(x,y)$$

$$e^{ax+by}\,\frac{1}{f(D+a, D'+b)}\,\psi(x,y) = \frac{1}{f(D,D')}\,e^{ax+by}\,\psi(x,y)$$

Replacing $\psi(x,y)$ by $\phi(x,y)$

$$\frac{1}{f(D,D')}\,e^{ax+by}\,\phi(x,y) = e^{ax+by}\,\frac{1}{f(D+a, D'+b)}\,\phi(x,y)$$

Note: If $f(a,b) = 0$

$$\frac{1}{f(D,D')}\,e^{ax+by}$$

is evaluted by $e^{ax+by} \cdot 1$

Example: solve

$$(D^2-D'^2-3D+3D')Z=xy+e^{x+2y}$$

solution

equation is equivalent to

$$(D-D')(D+D'-3)Z=xy+e^{x+2y}$$

$$C.F.=\phi_1(-x-y)+\phi_2(x-y)e^{3x}$$

$$P.I.=\frac{1}{(D-D')(D+D'-3)}xy+\frac{1}{(D-D')(D+D'-3)}e^{x+2y}$$

$$\frac{1}{(D-D')(D+D'-3)}xy=-\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)^{-1}xy$$

$$=-\frac{1}{3D}\left\{1+\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right\}\left\{1+\frac{D+D'}{3}+\frac{2DD'}{9}+\cdots\right\}xy$$

$$=-\frac{1}{3D}\left\{1+\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right\}\left\{xy+\frac{x+y}{3}+\frac{2}{9}\right\}$$

$$=-\frac{1}{3D}\left\{xy+\frac{1}{3}x+\frac{1}{3}y+\frac{2}{9}+\frac{x^2}{2}+\frac{1}{3}x\right\}$$

$$=-\frac{1}{3D}\left\{xy+\frac{2}{3}x+\frac{x^2}{2}+\frac{1}{3}y+\frac{2}{9}\right\}$$

$$=-\frac{1}{3}\left\{\frac{x^2y}{2}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{1}{3}xy+\frac{2}{9}x\right\}$$

$$\frac{1}{(D-D')(D+D'-3)}e^{x+2y}=\frac{1}{(D-D')(D+D'-3)}e^{x+2y}\cdot 1$$

$$=e^{x+2y}\frac{1}{(D+1-D'-2)(D+1+D'+2-3)}\cdot 1$$

$$= e^{x+2y} \frac{1}{(D+D'-1)(D+D')} \cdot 1$$

$$= e^{x+2y} \frac{1}{(D-D'-1)} e^{0x+0y} \frac{1}{(D+D')} \cdot 1$$

$$= e^{x+1y} \frac{1}{0-0-1} e^{0x+0y} \frac{1}{(D+D')} \cdot 1$$

$$= e^{x+2y} \frac{1}{(-1)(D+D')} \cdot 1$$

$$= e^{x+2y} \frac{-1}{D}\left(1+\frac{D'}{D}\right)^{-1} \cdot 1$$

$$= e^{x+2y} \frac{-1}{D}\left\{1-\frac{D'}{D}+\cdots\right\} \cdot 1$$

$$= e^{x+2y} \frac{-1}{D}(1) = -x e^{x+2y}$$

$$\therefore Z = \phi_1(-x-y) + \phi_2(x-y) e^{3x} - x e^{x+2y} - \frac{1}{6}x^2 y - \frac{1}{9}x^2 - \frac{1}{18}x^3 - \frac{1}{9}xy - \frac{2}{27}x.$$

Consider an equation

$$f(xD, yD')Z = V(x,y) \longrightarrow ①$$

where

$$f(xD, yD') = \sum_{r,s} C_{rs} x^r y^s D^r D'^s, \quad C_{rs} = \text{Const}.$$

Eq. ① is reduced to a linear Partial differential equation with constant co-efficients by the substitution

$u = \log x$ & $v = \log y \longrightarrow$ ②

$$xD = x\frac{\partial}{\partial x}$$
$$= x\frac{\partial}{\partial u}\frac{\partial u}{\partial x} = x\frac{\partial}{\partial u}\frac{1}{x}$$
$$= \frac{\partial}{\partial u} = d$$
$$x^2D^2 = x^2D(D)$$
$$= x^2D\left(\frac{1}{x}\frac{\partial}{\partial u}\right)$$
$$= x^2\left\{-\frac{1}{x^2}\frac{\partial}{\partial u} + \frac{1}{x}\frac{\partial^2}{\partial u^2}\frac{\partial u}{\partial x}\right\}$$
$$= x^2\left\{-\frac{1}{x^2}\frac{\partial}{\partial u} + \frac{1}{x^2}\frac{\partial^2}{\partial u^2}\right\}$$
$$= \frac{\partial^2}{\partial u^2} - \frac{\partial}{\partial u}$$
$$= \frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\right) - \frac{\partial}{\partial u}$$
$$= \frac{\partial}{\partial u}\left\{\frac{\partial}{\partial u} - 1\right\}$$
$$= d(d-1)$$

In general

$$x^rD^r = d(d-1)(d-2)\dots(d-\overline{r-1})$$

similarly

$$y^sD'^s = d'(d'-1)(d'-2)\dots(d'-\overline{s-1})$$

$\therefore f(xD, yD') = \sum c_{rs}\, d(d-1)\ldots(d-\overline{r-1})\ldots$
$d'(d'-1)\ldots(d'-\overline{s-1})$
$= g(d, d')$

where co-efficients in $g(d,d')$ are constants.

Thus by substituting ②, eq. ① is reduced to

$$g(d,d')Z = v(e^u, e^v)$$
$$= U(u, v)$$

Example: Solve

$$(x^2D^2 - y^2D'^2 - yD' + xD)Z = 0 \longrightarrow ①$$

Solution

Put $u = \log x$ & $v = \log y$

Eq. ① is reduced to

$$(d(d-1) - d'(d'-1) - d' + d)Z = 0$$
$$\{d^2 - d - d'^2 + d' - d' + d\}Z = 0$$
$$\{d^2 - d'^2\}Z = 0$$

A.E. is

$$m^2 - 1 = 0$$
$$m = 1, -1$$
$$Z = \phi_1(v+u) + \phi_2(v-u)$$
$$Z = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$$
$$Z = \phi_1(\log xy) + \phi_2\left(\log\left(\frac{y}{x}\right)\right)$$
$$Z = \psi_1(xy) + \psi_2\left(\frac{y}{x}\right).$$

Partial differential equations of order two with variable co-efficients

second order Partial derivatives will be denoted by

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad t = \frac{\partial^2 z}{\partial y^2}$$

An equation is said to be of order two, if it involves at least one of differential co-efficients r, s, t, but none of higher order; the quantities p and q may also enter into the equation.

The general form of equation of order two is

$$f(x, y, z, p, q, r, s, t) = 0 \longrightarrow (1)$$

Eq. (1) is linear with respect to dependent variable z and its derivatives it must be of the form

$$Rr + Ss + Tt + Pp + Qq + Zz = F \longrightarrow (2)$$

where R, S, T, P, Q, Z and F may be functions of x and y only and not all R, S, T are zero.

Different types of eq. ② will be considered.

Type 1:

$$Rr = F$$
$$Ss = F$$
$$Tt = F$$

$$\left.\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{F}{R} = F_1(x,y) \\ s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{F}{S} = F_2(x,y) \\ t &= \frac{\partial^2 z}{\partial y^2} = \frac{F}{T} = F_3(x,y) \end{aligned}\right\} \longrightarrow ③$$

Eqs. ③ are reducible equations with constant co-efficients.

Example: Solve $r = \sin(xy)$

Solution
$$\frac{\partial^2 z}{\partial x^2} = \sin(xy) \longrightarrow ①$$

integrating w.r.t. x

$$\frac{\partial z}{\partial x} = -\frac{1}{y}\cos(xy) + \phi_1(y) \longrightarrow ②$$

where ϕ_1 is an arbitrary function.

Integrating eq. ② w.r.t. x

$$z = -\frac{1}{y^2}\sin(xy) + \phi_1(y)\,x + \phi_2(y)$$

where ϕ_2 is an arbitrary function.

Type 2:

$$Rr + Pp = F(x,y)$$
$$Ss + Pp = F(x,y)$$
$$Ss + Qq = F(x,y)$$
$$Tt + Qq = F(x,y)$$

$$\left.\begin{array}{l} R\dfrac{\partial p}{\partial x} + Pp = F \\ S\dfrac{\partial p}{\partial y} + Pp = F \\ S\dfrac{\partial q}{\partial x} + Qq = F \\ T\dfrac{\partial q}{\partial y} + Qq = F \end{array}\right\} \longrightarrow (1)$$

ordinary differential equations which are linear of order one in which p (or q) is the dependent variable.

Example: Solve $xr + p = 9x^2y^2$

Solution

$$x\frac{\partial^2 z}{\partial x^2} + p = 9x^2y^2$$

$$x\frac{\partial}{\partial x}\frac{\partial z}{\partial x} + p = 9x^2y^2$$

$$x\frac{\partial p}{\partial x} + p = 9x^2y^2$$

$$\frac{\partial p}{\partial x} + \frac{1}{x}p = 9xy^2$$

$$P = e^{-\int \frac{1}{x}dx}\left[\phi_1(y) + \int e^{\int \frac{1}{x}dx} 9xy^2\,dx\right]$$

$$P = e^{-\ln x}\left[\phi_1(y) + \int e^{\ln x} 9xy^2\,dx\right]$$

$$P = \frac{1}{x}\left[\phi_1(y) + \int 9x^2y^2\,dx\right]$$

$$Px = \phi_1(y) + 3x^3y^2$$

$$P = \frac{1}{x}\phi_1(y) + 3x^2y^2$$

$$\frac{\partial Z}{\partial x} = \frac{1}{x}\phi_1(y) + 3x^2y^2$$

where ϕ_1 is an arbitrary function
integrating w.r.t. x

$$Z = \ln x\,\phi_1(y) + x^3y^2 + \phi_2(y)$$

where ϕ_2 is an arbitrary function.

Type 3 :

$$Rr + Ss + Pp = F$$
$$Ss + Tt + Qq = F$$

$$\left.\begin{aligned} R\frac{\partial p}{\partial x} + S\frac{\partial p}{\partial y} &= F - Pp \\ S\frac{\partial q}{\partial x} + T\frac{\partial q}{\partial y} &= F - Qq \end{aligned}\right\} \text{(1)}$$

Eqs. (1) are linear with p (or q) as dependent variable and x, y as independent variables.

Example: Solve $r + \frac{y}{x} s = 15xy^2$

Solution

$$\frac{\delta^2 z}{\delta x^2} + \frac{y}{x} \frac{\delta^2 z}{\delta x \delta y} = 15xy^2$$

$$\frac{\delta P}{\delta x} + \frac{y}{x} \frac{\delta P}{\delta y} = 15xy^2$$

Lagrange's subsidiary eqs. are

$$\frac{dx}{1} = \frac{dy}{\frac{y}{x}} = \frac{dP}{15xy^2}$$

$$\frac{y}{x} dx = dy$$

$$\frac{dx}{x} = \frac{dy}{y}$$

integrating $\ln x + \ln a = \ln y$

$$ax = y$$

$$\frac{y}{x} = a \quad , \; a - \text{Const}$$

$$dP = 15xy^2 dx$$

$$dP = 15x(ax)^2 dx$$

$$dP = 15a^2 x^3 dx$$

integrating

$$P = 15a^2 \frac{x^4}{4} + \text{Const}$$

$$P = \frac{15}{4}(ax)^2 x^2 + \text{Const} \quad , \; y = ax$$

$$p = \frac{15}{4} x^2 y^2 + \phi\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x} = \frac{15}{4} x^2 y^2 + \phi\left(\frac{y}{x}\right)$$

integrating w.r.t. x

$$Z = \frac{5}{4} x^3 y^2 + \int \phi\left(\frac{y}{x}\right) dx + \phi_1(y)$$

$$\frac{y}{x} = \text{const} \longrightarrow \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2 = \text{const}$$

$$Z = \frac{5}{4} x^3 y^2 + \int \phi\left(\frac{y}{x}\right) \frac{1}{-\frac{y^2}{x^2}} \cdot d\left(\frac{y}{x}\right) \cdot y + \phi_1(y)$$

$$Z = \frac{5}{4} x^3 y^2 + y \int \frac{1}{\left(-\frac{y^2}{x^2}\right)} \phi\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right) + \phi_1(y)$$

$$Z = \frac{5}{4} x^3 y^2 + y \phi_2\left(\frac{y}{x}\right) + \phi_1(y)$$

where ϕ, ϕ_1 and ϕ_2 are arbitrary functions.

Type 4 : $\left.\begin{aligned} Rr + Pp + Zz = F \\ Tt + Qq + Zz = F \end{aligned}\right\} \longrightarrow (1)$

Eqs. (1) are linear ordinary differential equations with x (or y) as independent variable and z as dependent variable.

Example: Solve

$$r - p - \frac{1}{y}\left(\frac{1}{y} - 1\right)z = x^2y - x^2y^2 + 2xy^3 - 2y^3 \longrightarrow ①$$

Solution

$$\left\{D^2 - D - \frac{1}{y}\left(\frac{1}{y} - 1\right)\right\}z = x^2y - x^2y^2 + 2xy^3 - 2y^3$$

$$\left(D - \frac{1}{y}\right)\left(D + \left(\frac{1}{y} - 1\right)\right)z = x^2y - x^2y^2 + 2xy^3 - 2y^3$$

$$C.F. = \phi_1(-y)\, e^{\frac{x}{y}} + \phi_2(-y)\, e^{x - \frac{x}{y}}$$

To obtain P.I., assume

$$z = Ax^2 + Bx + C \longrightarrow ②$$

where A, B and C are functions of y or consts.

$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = 2Ax + B \\ r &= \frac{\partial^2 z}{\partial x^2} = 2A \end{aligned}\right\} \longrightarrow ③$$

Substituting from ② and ③ in ①

$$2A - 2Ax - B - \frac{1}{y}\left(\frac{1}{y} - 1\right)(Ax^2 + Bx + C) = x^2y - x^2y^2 + 2xy^3 - 2y^3$$

$$2A - B - \frac{1}{y}\left(\frac{1}{y} - 1\right)C + \left(-2A - \frac{1}{y}\left(\frac{1}{y} - 1\right)B\right)x - \frac{1}{y}\left(\frac{1}{y} - 1\right)Ax^2 = 2y^3x + (y - y^2)x^2 - 2y^3$$

Equating coefficients

$$2A - B - \frac{1}{y}\left(\frac{1}{y} - 1\right)C = -2y^3 \longrightarrow (4)$$

$$-2A - \frac{1}{y}\left(\frac{1}{y} - 1\right)B = 2y^3 \longrightarrow (5)$$

$$-\frac{1}{y}\left(\frac{1}{y} - 1\right)A = y - y^2 \longrightarrow (6)$$

From eq. (6)

$$\left(\frac{-1}{y^2} + \frac{1}{y}\right)A = y - y^2$$

$$\left(\frac{y-1}{y^2}\right)A = -y(y-1)$$

$$\frac{A}{y^2} = -y$$

$$A = -y^3$$

Then from (5)

$$2y^3 - \frac{1}{y}\left(\frac{1}{y} - 1\right)B = 2y^3$$

$$-\frac{1}{y}\left(\frac{1}{y} - 1\right)B = 0$$

$$B = 0$$

and from (4)

$$-2y^3 - \frac{1}{y}\left(\frac{1}{y}-1\right)C = -2y^3$$

$$-\frac{1}{y}\left(\frac{1}{y}-1\right)C = 0$$

$$C = 0$$

$$z = Ax^2 + Bx + C$$

$z = -x^2y^3$ is P.I.

$$\therefore z = e^{\frac{x}{y}}\phi_1(-y) + e^{x-\frac{x}{y}}\phi_2(-y) - x^2y^3$$